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THEORETICAL MECHANICS
STATICS

Tutorial

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PREFACE

INTRODUCTION

The progress of technology confronts the engineer with a wide variety of problems connected with the design, manufacture and operation of various machines, motors and structures. Despite the diversity of problems that arise, their solution at least in part, is based on certain general principles common to all of them, namely, the laws governing the motion and equilibrium of material bodies.

The science, which treats of the general laws of motion and equilibrium of material bodies, is called theoretical, or general, mechanics. Theoretical mechanics constitutes one of the scientific bedrocks of modern engineering.

By motion in mechanics we mean mechanical motion, i.e., any change in the relative positions of material bodies in space which occurs in the course of time.

According to the nature of the problems treated, mechanics is divided into statics, kinematics, and dynamics. Statics studies the forces and the conditions of equilibrium of material bodies subjected to the action of forces. Kinematics deals with the general properties of the motion of bodies. Dynamics studies the laws of motion of material bodies under the action of forces.

According to the nature of the objects under study, theoretical mechanics is subdivided into: mechanics of a particle, i. e., of a body whose dimensions can be neglected in studying its motion or equilibrium, and systems of particles; mechanics of a rigid body, i. e., a body whose deformation can be neglected; mechanics of bodies of variable mass; mechanics of deformable bodies; mechanics of liquids; mechanics of gases.

1. STATICS

1.1. BASIC CONCEPTS AND PRINCIPLES OF STATICS

1.1.1. The Subject of Statics

Statics is the branch of mechanics which studies the laws of composition of forces and the conditions of equilibrium of material bodies under the action of forces.

Equilibrium is the state of rest of a body relative to other material bodies. General mechanics deals essentially with equilibrium of solids.

All solid bodies change the shape to a certain extent when subjected to external forces. This is known as deformation. In order to ensure the necessary strength of engineering structures and elements, the material and dimensions of various parts are chosen in such a way that the deformation under specified loads would remain tolerably small.

This makes it possible, in studying the general conditions of equilibrium, to treat solid bodies as undeformable or absolutely rigid, ignoring the small deformations that actually occur. A perfectly rigid body is said to be one in which the distance between any pair of particles is always constant. In solving problems of statics bodies are considered as perfectly rigid.

For a rigid body to be in equilibrium when subjected to the action of a system of forces, the system must satisfy certain conditions of equilibrium. The determination of these conditions is one of the principal problems of statics. In order to find out the equilibrium conditions and to solve other problems one must know the principles of the composition of forces, the principles of replacing one force system by another and, particularly, the reduction of a given force system to as simple a form as possible. Accordingly, statics of rigid bodies treats of two basic problems:

1) the composition of forces and reduction of force system to as simple a form as possible, and

2) the determination of the conditions for the equilibrium of force system acting on rigid bodies.

1.1.2. Force

The state of equilibrium or motion of a given body depends on its mechanical interaction with other bodies. *The quantitative measure of the mechanical interaction of material bodies is called force.* Force is a vector quantity. Its action on a body is characterized by its magnitude, direction, and point of application.

We shall call any set of forces acting on rigid body a force system. We shall also use the following definitions:

A body not connected with other bodies and which from any given position can be displaced in any direction in space is called a free body.

If a force system acting on a free rigid body can be replaced by another force system without disturbing the body's initial condition of rest or motion, the two systems are said to be equivalent.

If a free rigid body can remain at rest under the action of a force system, that system is said to be balanced or equivalent to zero.

A resultant is a single force capable of replacing the action of a system of forces on rigid body. A force equal in magnitude, collinear with, and opposite in direction to the resultant is called an equilibrant force.

Forces acting on a rigid body can be divided into two groups: the external and internal forces. External forces represent the action of other material bodies on the particles of a given body. Internal forces are those with which the particles of a given body act on each other.

A force applied to one point of a body is called a concentrated force. Forces acting on all the points of a given volume or given area of a body are called distributed force. A concentrated force is a purely notional concept, insofar as it is actually impossible to apply a force to a single point of a body.

1.1.3. Axioms of Statics

There are some fundamental principles in statics which are called axioms. Some of these principles are corollaries of the fundamental laws of dynamics.

Axiom 1. *A free rigid body subjected to the action of two forces can be in equilibrium if, and only, if the two forces are equal in magnitude, collinear, and opposite in direction.*

Since we know that a free body subjected to the action of a single force cannot be in equilibrium, the first axiom defines the simplest balanced force system.

Axiom 2. *The action of given force system on a rigid body remains unchanged if another balanced force system is added to, or subtracted from, the original system*.

It follows that two force systems differing from each other by a balanced system are equivalent.

Corollary. *The point of application of a force, acting on a rigid body, can be transferred to any other point on the line of action of the force without altering its effect.*

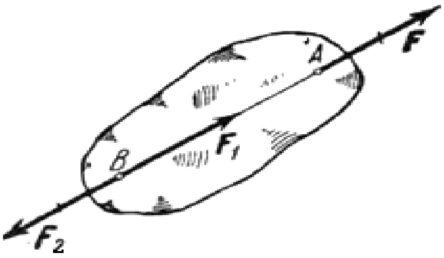


Fig.1.1.1

Consider a rigid body with a force $\boldsymbol{F}$ applied at a point *A* (Fig.1.1.1). In accordance with the Axiom 2 we can apply to the arbitrary point *B* on the line of action of the force $\boldsymbol{F}$ a balanced system $\boldsymbol{F}_1$, $\boldsymbol{F}_2$such that $\boldsymbol{F}_1 = \boldsymbol{F}$ and $\boldsymbol{F}_2 = -\boldsymbol{F}$. From the axiom 1 it follows that forces $\boldsymbol{F}$ and $\boldsymbol{F}_2$ also form a balanced system and cancel each other.

Thus, we have only force $\boldsymbol{F}_1$, equal to $\boldsymbol{F}$ in magnitude and direction, with the point of application shifted to point *B.*

It should be noted that this corollary holds good only for forces acting on perfectly rigid bodies.

Axiom 3. *Two forces applied at one point of a body have as the resultant a force applied at the same point and represented by the diagonal of a parallelogram constructed with the two given forces as its sides (Fig.1.1.2).*

It is well known that vector $\boldsymbol{R}$ is called the geometrical sum of the vectors $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$:

$$\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2.$$

Hence, the axiom 3 can also be formulated as follows:

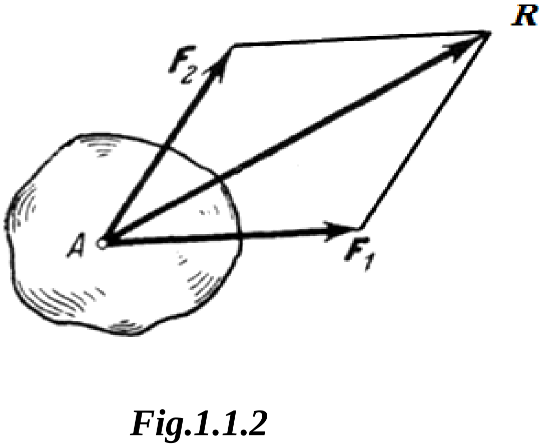


Fig.1.1.2

the resultant of two forces applied at one point of a body is the geometrical sum of those forces and is applied at that point.

It is very important to discriminate between concepts of a sum of forces and their resultant.

Axiom 4. *To any action of one material body on another there is always an equal and oppositely directed reaction.*

This axiom represents the third law of the dynamics. The law of action and reaction is one of the fundamental principles of mechanics.

It follows from it that when a body *A* acts

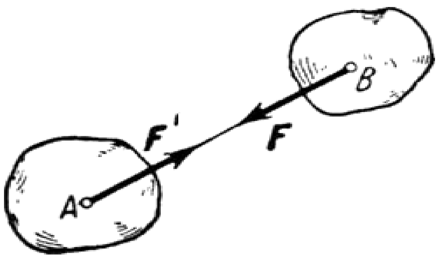


Fig.1.1.3

on a body B with a force $\boldsymbol{F}$, body B simultaneously acts on body A with a force $\boldsymbol{F}'$, equal in magnitude, collinear with, and opposite in sense to force $\boldsymbol{F}(\boldsymbol{F}' = -\boldsymbol{F})$ (Fig.1.1.3).

Axiom 5. If a freely deformable body subjected to the action of forces is in equilibrium, the state of equilibrium will not be disturbed if the body solidifies (becomes rigid).

This axiom which is called principle of solidification can also be formulated as follows: if a deformable body is in equilibrium, the forces acting on it satisfy the conditions for the equilibrium of rigid body.

The axiom of solidification is widely employed in engineering problems. It makes it possible to determine equilibrium conditions by treating a deformable body or structure as a rigid one and to apply to it the methods of rigid - body statics.

1.1.4. Constraints and Their Reactions

A body whose displacement in space is restricted by other bodies either connected to or in contact with it is called a constrained body. We shall call a constraint anything that restricts the displacement of a given body in space.

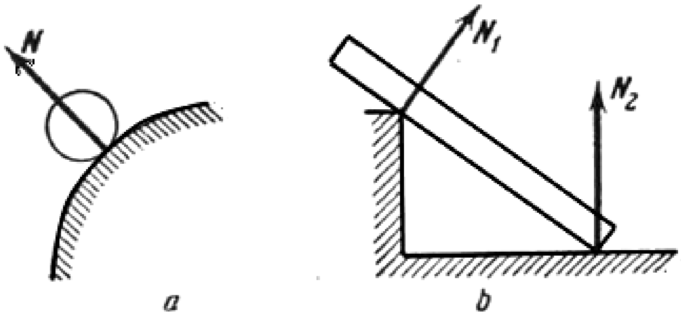
A body acted upon by a force or forces whose displacement is restricted by a constraint acts on that constraint with a force which is called the load or pressure acting on that constraint. At the same time, according to the axiom 4, the constraint reacts with a force of the same magnitude and opposite sense. The force, with which a constraint acts on a body thereby restricting its displacement, is called the force of reaction of the constraint, or simply the reaction of the constraint.

All forces which are not the reactions of constraints are called applied or active forces. The magnitude and direction of active forces do not depend on the other forces acting on a given body. The difference between a force of constraint and an active force is that the magnitude of the former always depends on the active forces and is not therefore immediately apparent.

The reactions of constraints are determined by solving corresponding problems of statics. The reaction of a constraint points away from the direction in which the given constraint prevents a body's displacement.

The correct determination of the direction of reactions is of great importance in solving problems. Let us therefore consider the direction of reactions of some common types of constraints.

1. Smooth surface or support. A smooth surface is one whose friction can be neglected in the first approximation. Such a surface prevents the displacement of a body perpendicular (normal) to both contacting surfaces at their point of contact



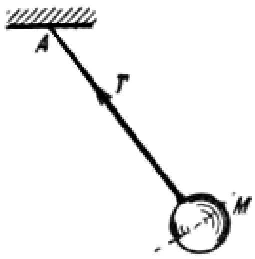


Fig.1.1.4 ***Fig.1.1.5***

Therefore, the action of a smooth surface or support is directed normal to both contacting surfaces at their point of contact and is applied at that point (Fig.1.1.4.).

If one of the contacting surfaces is a point then the reaction is directed normal to the other surface (fig.1.1.4b).

2. String. A constraint provided by a flexible inextensible string (Fig.1.1.5.) prevents a body M from receding from the point of suspension of the string in the direction AM. The reaction $\boldsymbol{T}$ of the string is thus directed along the string towards the point of suspension.

3.Cylindrical Pin (Bearing). When two bodies are joined by means of a pin passing through holes in them, the connection is called a pin joint or hinge. Body

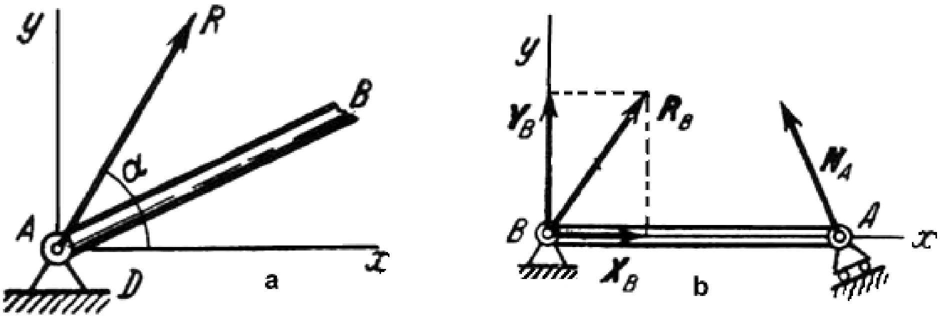


Fig.1.1.6

AB in Fig.1.6a is hinged to support D and can rotate freely in the plane of the figure about the axis of the joint. At the same time, point A cannot be displaced in any direction perpendicular to the axis. Thus, the reaction $\boldsymbol{R}$ of a pin can have any direction in the plane perpendicular to the axis of the joint (Fig.1.1.6a).

It is very important to discriminate between bearings and moved cylindrical pin (roller support). The main difference is that in moved cylindrical pin its axis can move along fixed plane. Thus, its reaction is normal to this fixed plane (Fig.1.1.6b).

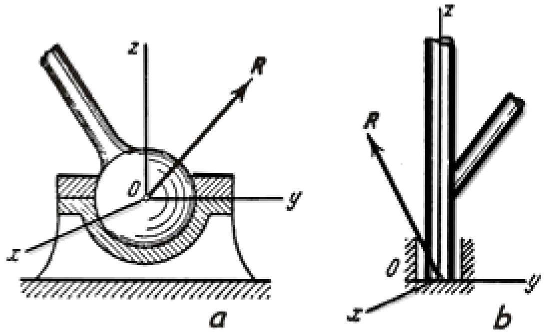


Fig.1.1.7

4. Ball - and - Socket joint, step bearing. This type of the constraint prevents

displacements in any direction (Fig.1.1.7). Examples of such a constraint are a ball - pivot (Fig.1.1.7*a*) and a step bearing (Fig.1.1.7b). The reaction $\boldsymbol{R}$ of a ball - and – socket joint or step bearing has any direction in space. Neither magnitude of $\boldsymbol{R}$ nor its direction in space is immediately apparent.

5. Rod. Let a rod of a negligible weight and secured by hinges at its ends be the constraint of a certain structure. Then only two forces applied at its ends act on the rod. If rod is in equilibrium, the forces, according to the axiom 1, must be collinear and directed along the axis of the rod. Consequently, a rod subjected to forces applied at its tips, where the weight of the rod is negligible, can be only under tension or under compression, i. e., reaction of a rod is directed along its axis.

Finally, let's consider the axiom of constraints which permits to reduce the problems of equilibrium of constrained bodies to study of free ones:
any constrained body can be treated as a free body relieved from its constraints, provided the latter are represented by their reactions.

1.2. CONCURRENT FORCE SYSTEM

1.2.1. Composition of Forces. Resultant of Concurrent Forces

We shall commence our study of statics with the geometric method of composition of forces. As forces are vector quantities their addition and resolution are based on the laws of vector algebra. The quantity which is the geometric sum of all the forces of a given system is called the principal vector of the system. The concept of the geometric sum of forces should not be confused with that of resultant. As we shall see later on, many force systems have no resultant at all.

The geometric sum R of two concurrent forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ is determined either by the parallelogram rule or by constructing a force triangle.

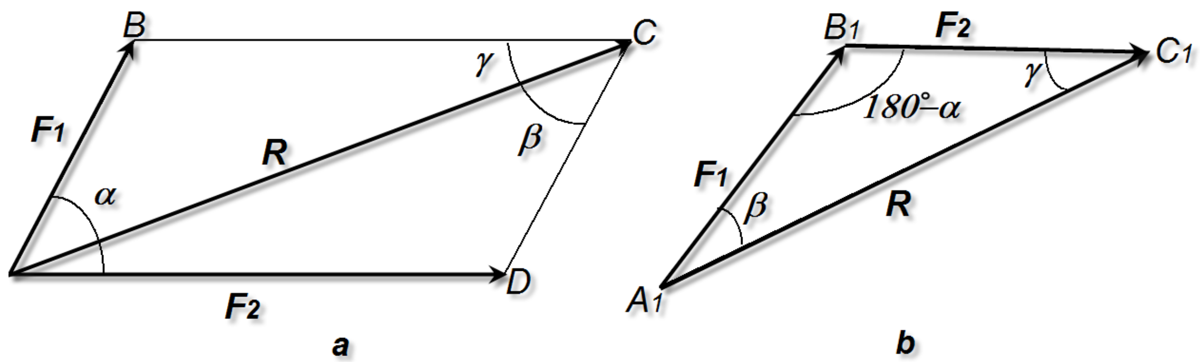
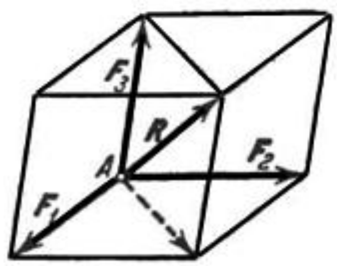


Fig.1.2.1

The magnitude of $\boldsymbol{R}$ is the side A_1C_1 of the triangle $A_1B_1C_1$, i.e.,
where α is the angle between the two forces.

The angles β and γ which the resultant R makes with the component forces can be determined by the laws of sins:

$$\frac{F_1}{\sin\gamma} = \frac{F_2}{\sin\beta} = \frac{R}{\sin\alpha}.$$



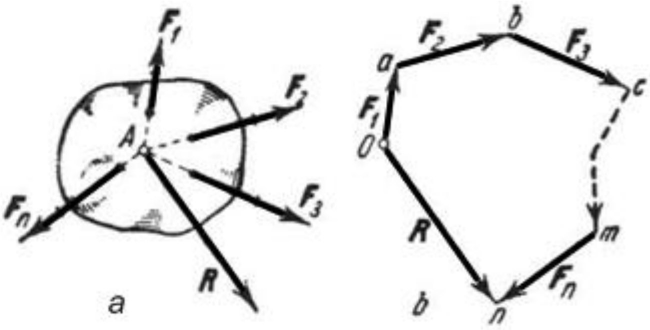


Fig.1.2.2. *Fig.1.2.3.*

The geometrical sum of three non - coplanar forces is represented by the diagonal of a parallelepiped with the given forces for its edge (Fig.1.2.2). This rule can be verified by successively applying the parallelogram rule.

The geometrical sum of any forces can be determined by constructing a force polygon (Fig.1.2.3). The magnitude and direction of R do not depend on the order in which the vectors are laid off. The figure constructed in Fig.1.2.3b is called a force polygon or vector polygon. Thus, the geometrical sum, or principal vector, of a set of forces is represented by a closing side of a force polygon constructed with the given forces as its sides.

Forces whose lines of action intersect at one point are called concurrent. Consecutively applying the parallelogram rule, we come to the conclusion that the resultant of a system of concurrent forces is equal to the principal vector of those forces and that it is applied at the point of intersection of these forces.

To resolve a force into two or more components means to replace it by a force system whose resultant is the original force. Resolution of forces like their addition is based on the rules of vector algebra.

1.2.2. Projection of a Force on an Axis and on a Plane

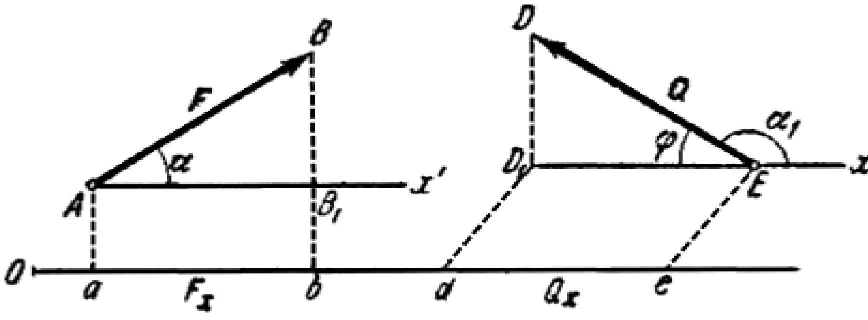


Fig.1.2.4.

Analytical methods of solving problems of statics are based on the concept of the projection of a force on an axis. The projection of a force on an axis is an algebraic quantity equal to the length of the line segment comprised between the projections of the initial and terminal points of the force taken with the appropriate sign. We shall take "plus" if the direction from the initial to the terminal point is the positive direction of the axis, and "minus" if it is the negative direction of the axis

(Fig.1.2.4). It follows from this definition that the projections of a given force on any parallel axes of same sense are equal. We shall denote the projection of a force $\boldsymbol{F}$ on an axis Ox by the symbol F_x .

It is apparent from the Fig.1.2.4 that

$$F_x = F\cos\alpha. \qquad (1.2.1)$$

Hence, the projection of a force on an axis is equal to the product of the magnitude of the force and the cosine of the angle between the direction of the force and the positive direction of the axis. It follows from formula (1.2.1) that

$F_x > 0$ if $\alpha < \pi/2,\ \alpha < \pi/2, F_x < 0$ if $\alpha > \pi\ /\ 2$ and $F = 0_x$ if $\alpha = \pi\ /\ 2$.

The projection of a force on a plane is a vector comprised between the projections of the initial and terminal points of the force on the plane (Fig. 1.2.5).

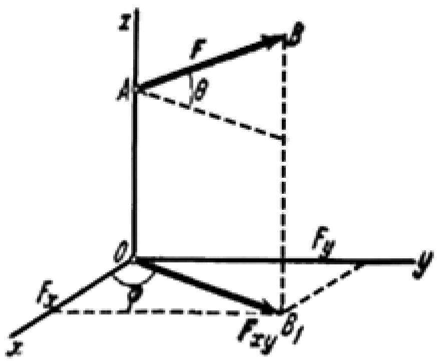


Fig.1.2.5

Thus, unlike the projection of a force on an axis the projection of a force on a plane is vector quantity. The magnitude of the projection is $F_{xy} = F\cos\theta$, where θ is the angle between the direction of force $\boldsymbol{F}$ and its projection F_{xy}. It also follows from Fig. 1.2.5 that

$$F_x = F_{xy}\cos\varphi = F\cos\theta\cos\varphi$$
$$F_y = F_{xy}\sin\varphi = F\sin\theta\cos\varphi.$$

1.2.3. Analytical Method of Defining and Composition of Forces

Let's select a system of coordinate axes Oxy as a frame of reference for defining the direction of a force in space (Fig.1.2.6). For the solution of problems of statics it is convenient to define a force by its projections, i.e. any force $\boldsymbol{F}$ is completely defined if its projections F_x, F_y and F_z on the axes of a coordinate system are known. From formula (1.2.1) we have

$$F_x = F\cos\alpha\ , F_y = F\cos\beta,\ F_z = F\cos\gamma.$$

We can obtain from these equations

$F_x^2 + F_y^2 + F_z^2 = F^2$, since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$,

whence,

$$F_x^2 + F_y^2 + F_z^2 = F^2\ ,\ \cos\alpha = F_x/F\ ,\ \cos b = F_y/F,\ \cos\gamma = F_z/F\ . \qquad (1.2.2)$$

Eqs. (1.2.2) define the magnitude of a force and the angles it makes with the coordinate axes in terms of its projections on the given axes, i.e., they define the force.

If a set of given forces is coplanar, each force can be defined by its projections on two coordinate axes.

Operations with vectors can be expressed in terms of operations with their projections by the following theorem: *the projection of the vector of a sum on an axis is equal to the algebraic sum of the projections of the component vectors on the same axis.*

From this theorem we obtain that for any force system $F_1, \boldsymbol{F}_2 \ldots \boldsymbol{F}_n$ whose principal vector is $\boldsymbol{R} = \sum_k \boldsymbol{F}_k$, we have

$$R_x = \sum_k F_{kx}, R_y = \sum_k F_{ky}, \; R_z = \sum_k F_{kz} \quad (1.2.3)$$

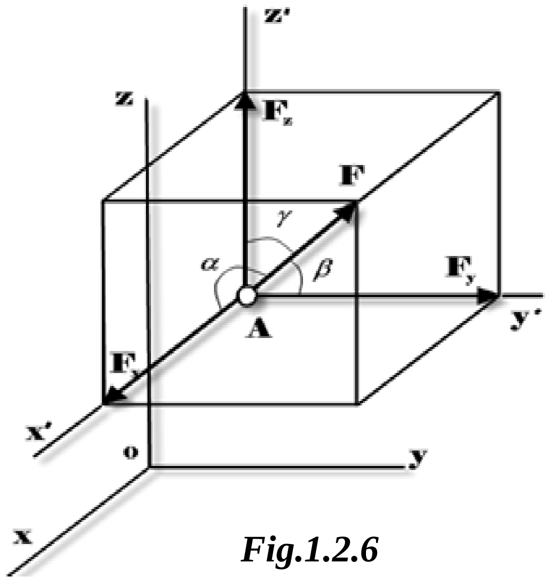


Fig.1.2.6

Knowing R_x, R_y and R_z, from formulas (1.2.3) we obtain

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}, \quad (1.2.4)$$

$$\cos\alpha = {R_x}/{R}, \; \cos\beta = {R_y}/{R},$$

$$\cos\gamma = {R_2}/{R}.$$

Eqs. (1.2.3) and (1.2.4) provide an analytical solution of the composition of forces.

1.2.4. Conditions for the Equilibrium of a Concurrent Force System

It follows from the laws of mechanics that a rigid body under the action of set of mutually balanced forces can either be at rest or in motion. We call this kind of motion "inertial motion". From this we can obtain two conclusions: a) forces acting on bodies at rest and on bodies in "inertial" motion equally satisfy the conditions of equilibrium; b) the equilibrium of forces acting on a free rigid body is a necessary but insufficient condition for the equilibrium of the body. The body will remain in equilibrium only if it was at rest before the moment when the balanced forces were applied.

For a system of concurrent forces to be in equilibrium it is necessary and sufficient for the resultant of the forces to be zero.

The conditions of equilibrium can be expressed in either graphical or in analytical form.

1) Graphical Condition of Equilibrium.

Since the resultant $\boldsymbol{R}$ of a concurrent force system is defined as the closing side of a force polygon, it follows that $\boldsymbol{R}$ can be zero only if the polygon is closed.

Thus, *for a system of concurrent forces to be in equilibrium it is necessary and sufficient for the force polygon drawn with these forces to be closed.*

2) Analytical Conditions of Equilibrium.

The resultant of a concurrent force system is determined by the formula (1.2.4). As the expression under the radical is a sum of positive components, R can be zero only if $R_x = R_y = R_z = 0$ simultaneously, i.e., taking into account Eqs. (1.2.3) if

$$\sum_k F_{kx} = 0, \quad \sum_k F_{ky} = 0, \quad \sum_k F_{kz} = 0. \qquad (1.2.5)$$

Thus, *for a system of concurrent forces to be in equilibrium it is necessary and sufficient for the sums of the projections of all forces on each of the coordinate axes to be zero.*

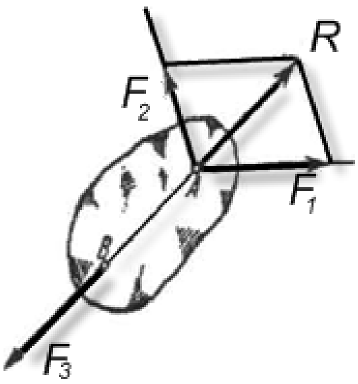


Fig.1.2.7

There are only two equations in (1.2.5) if the concurrent forces form coplanar system.

3) The Theorem of Three Forces.

It is often very useful the following theorem: *if a free rigid body remains in equilibrium under the action of three nonparallel coplanar forces, their lines of action intersect at one point.*

First draw two of the forces acting on the body, say $\boldsymbol{F}_1$ (Fig.1.2.7) and $\boldsymbol{F}_2$. As the theorem states, their lines of action intersect at some point A. Now replace them by their resultants. Two forces will be acting on the body: $\boldsymbol{R}$ and F_3, which is applied at some point B.

If the body is to be in equilibrium, then, according to the axiom 1, forces $\boldsymbol{R}$ and F_3 must be directed along the same line, i.e., along AB. Consequently, force F_3 also passes through A, and the theorem is proved. It should be noted that the reverse is not true, i.e., the theorem expresses a necessary, but not sufficient, condition for the equilibrium of a body acted upon by three forces.

1.2.5. Illustrative Problems

The solution of problems, as a rule, consists of the following steps:

1. Choose the body whose equilibrium should be examined. For the problem to lend itself to solution, the given and required forces, or their equivalents, should all be applied to the body whose equilibrium is being examined (for instance, if the problem is to determine a load acting on a support, we can examine the equilibrium of the body experiencing the reaction of the support, which is equal in magnitude to the required load).

If the given forces act on one body and the required on another, it may be necessary to examine the equilibrium of each body separately, or even of some intermediary bodies as well

2. Isolate the body from its constraints and draw the given forces and the reactions of the removed constraints. Such a drawing is called a free-body diagram (FBD) and is drawn separately.

3. State the conditions of equi1ibrium. The statement of these conditions depends on the force system acting on the free body and the method of solution (graphical or analytical). Special cases of stating the equilibrium conditions for different force systems will be examined in the respective chapters of this book.

4. Determine the unknown quantities, verify the answer and analyze the results. The computations should, as a rule, be written out in general (algebraic) form. This provides formulas for determining the unknown quantities, which can then be used to analyze the results. Solution in general form also makes it possible to catch mistakes by checking the dimensions (the dimensions of the terms in each side of an

equation should be the same). If the problem is solved in general form, the numerical values should be substituted in the final equations.

In this section, we shall discuss equilibrium problems involving concurrent forces. They can be solved by either the graphical or the analytical method.

The graphical method is suitable when the total number of given and required forces acting on a body is three. If the body is in equilibrium the force triangle must be closed (the construction should start with the known force). By solving the triangle trigonometrically, we obtain the unknown quantities.

The analytical method can be applied for any number of forces. Before writing the conditions of equilibrium the coordinate axes must be chosen. The choice is arbitrary, but the equations can be simplified by taking one of the axes perpendicular to an unknown force.

The method of resolution of forces is useful in determining the pressure on constraints induced by applied forces. Loads acting on rigid constraints are determined by resolving the given forces along the directions of the reactions of the constraints as, according to the 4th principle, force acting on a constraint and its reaction have the same line of action. It follows, then, that this method can be applied only if the directions of the reactions of the respective constraints are immediately apparent.

Problem 1. Members AC and BC of the bracket in Fig.1.2.8a are joined together and attached to the wall with pins. Neglecting the weight of the members, determine the thrust in BC if the suspended load weighs $P, \angle BAC = 90°$, and $\angle ABC = \alpha$.

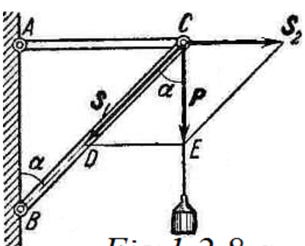


Fig.1.2.8.a

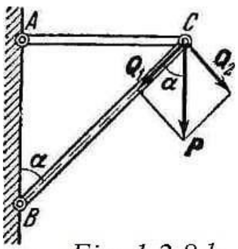


Fig. 1.2.8 b

Solution. Force $\boldsymbol{P}$ acts on both members, and the reactions are directed along them. The unknown thrust is determined by applying force $\boldsymbol{P}$ at point C and resolving it along AC and BC. Component $\boldsymbol{S}_1$ is the required force. From triangle CDE we obtain:

$$S_1 = \frac{P}{\cos\alpha}.$$

From the same triangle we find that member AC is under a tension of $S_2 = P\tan\alpha$.

The larger the angle , the greater the load on both members, which increases rapidly as approaches 90°. For example, at $P = 100\ kN$ and $\alpha = 85°$ $S_1 \approx 1150\ kN$ and $S_2 \approx 1140kN$. Thus, to lessen the load angle α should be made smaller.

We see from these results that a small applied force can cause very large stresses in structural elements (see also Problem 2). The reason for this is that forces are compounded and resolved according to the parallelogram law: a diagonal of a parallelogram can be very, much smaller than its sides. If, therefore, in solving a problem you find that the loads or reactions seem too big as compared with the applied forces, this does not necessarily mean that your solution is wrong.

Finally, beware of a mistake frequently made in applying the method of force resolution. In Problem 1 we have to determine the force of thrust acting on member

BC. If we were to apply force $\boldsymbol{P}$ at C (Fig.1.2.8b) and resolve it into a component $\boldsymbol{Q}_1$ along BC and a component $\boldsymbol{Q}_2$ perpendicular to it, we should obtain

$$Q_1 = P\cos\alpha, Q_2 = P\sin\alpha.$$

Although force $\boldsymbol{P}$ was resolved according to the rule, component $\boldsymbol{Q}_1$ is not the required force acting on BC because not all of force $\boldsymbol{Q}_2$ acts on AC. Actually force $\boldsymbol{Q}_2$ acts on both members and, consequently, it increases the load acting on BC and adds to $\boldsymbol{Q}_1$.

This example shows that if a force is not resolved along the reactions of the respective constraints the required result cannot be obtained.

Problem 2. A lamp of weight $P = 200N$ (Fig.1.2.9) hangs from two cables AC and BC forming equal angles $\alpha = 5°$ with the horizontal. Determine the tensions in the cables.

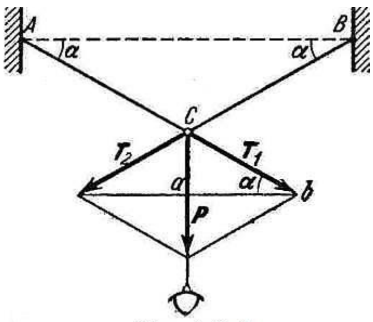


Fig. 1.2.9

Solution. Resolve force $\boldsymbol{P}$ applied at C into components directed along the cables. The force parallelogram in this case is rhombus whose diagonals are mutually perpendicular and bisecting. From triangle ACB we obtain

$$\frac{P}{2} = T_1\sin\alpha,$$

whence $T_1 = T_2 = \frac{P}{2\sin\alpha} \approx 1150\ N.$

The equation shows that the smaller the angle α the greater the tension in the cables (for instance, at α=*1*°, $T \approx 5730\ N$). Should we attempt to stretch the cable absolutely horizontally it would break, for, at $\alpha \to 0, T \to \infty$.

Problem 3. Neglecting the weight of rod AB and crank OB of the reciprocating gear in Fig.1.2.10, determine the circumferential force at B and the load on axle O of the crank caused by the action of force $\boldsymbol{P}$ applied to piston A if the known angles are α and β.

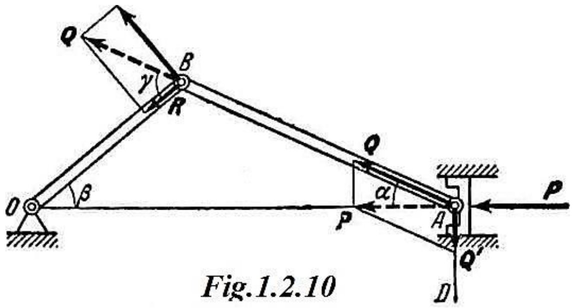


Fig.1.2.10

Solution. In order to determine the required forces we have to know the force $\boldsymbol{Q}$ with which the connecting rod AB acts on pin B. The magnitude of $\boldsymbol{Q}$ can be found by resolving force P along AB and perpendicular to OA. Thus we obtain: $Q = \frac{P}{\cos\alpha}$.

Transferring force Q to point B and resolving it as shown in Fig.1.2.10 into the circumferential force F and the load R on the axle, we obtain:

$$F = Q\sin\gamma, \qquad R = Q\cos\gamma.$$

Angle γ is an external angle of triangle *OBA* and equals $\alpha + \beta$. Hence, we finally obtain:

$$F = P\frac{\sin(\alpha+\beta)}{\cos\alpha}, \qquad R = P\frac{\cos(\alpha+\beta)}{\cos\alpha}.$$

As $\alpha + \beta \leq 180°$ and $\alpha < 90°$, force F is always greater than zero, i.e., it is always directed as shown in the Fig.1.2.10. Force R, however, is directed from B to

Oonly as long as $\alpha + \beta < 90°$; at $\alpha + \beta > 90°$, R reverses its sense. At $\alpha + \beta = 90°$, $R = 0$.

Problem 4. A load of weight P lies on a plane inclined at α degrees to the horizontal (Fig.1.2.11a). Determine the magnitude of the force F parallel to plane which should be applied to the load to keep it in equilibrium, and pressure **Q** exerted by the load on the plane.

Solution. The required forces act on different bodies: F on the load and **Q** on the plane. To solve the problem we shall determine instead of **Q** the reaction N of the plane, which is equal to **Q** in magnitude and opposite in sense. In this case the given force ***P*** and the required forces F and N all act on the load, i.e., on one body. Consider the equilibrium of the load as a free body (Fig.1.2.11b), with the applied forces ***P*** and F and reaction N of the constraint (the plane). The required forces can be determined by employing either the graphical or the analytical method.

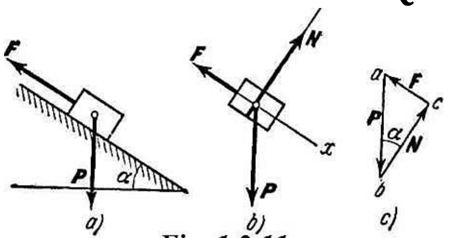


Fig. 1.2.11

Graphical method. If the body is in equilibrium, the force triangle with ***P***, ***N***, and F as its sides must be closed. Start the construction with the given force: from an arbitrary point *a* lay off to scale force ***P*** (Fig.1.2.11c).Through its initial and terminal points draw straight lines parallel to the directions of the forces F and ***N***. The intersection of the lines gives us the third vertex *c* of the closed force triangle abc, whose sides bc and ca denote the required forces in the chosen scale. The direction of the forces is determined by the arrow rule: as the resultant is zero, no two arrowheads can meet in any vertex of the triangle.

The magnitude of the required forces can also be computed (in which case the diagram need not be drawn to scale). Observing that angle $\angle abc = \alpha$ we have

$$F = P \sin \alpha, N = P \cos \alpha.$$

Analytical method. Since the force system is coplanar, only two coordinate axes are needed. To simplify the computation, take axis 0x perpendicular to the unknown force N.

From equilibrium equations, we obtain:

$$P \sin \alpha - F, -P \cos \alpha + N = 0,$$

which give:

$$F = P \sin \alpha, N = P \cos \alpha.$$

The force exerted by the load on the plane is equal in magnitude to the calculated force and opposite in sense.

It will be noted that the force **F** needed to hold the load on the inclined plane is less than its weight ***P***. Thus, an inclined plane represents a simple machine, which makes it possible to balance a large force with a smaller one.

A general conclusion can be drawn from the solution of the above problem: *in problems of statics solved by the equations of equilibrium, the forces exerted by a body on its constraints should be replaced by the reactions of the constraints acting on the body, which are equal in magnitude and opposite in sense to the applied* *forces*. In solving problems by the method of force resolution the forces exerted by the constraints are determined directly.

Problem 5. The rod AB in Fig.1.2.12a is hinged to a fixed support at A. Attached to the rod at B is a load $P = 100\ N$ and a string passing over a pulley at C with a load $Q = 141\ N$ tied to the other end of the string. The axes of the pulley C and the pin A lie on the same vertical and AC = AB. Neglecting the weight of the rod and the diameter of the pulley, determine the angle α at which the system will be in equilibrium and the stress in the rod AB.

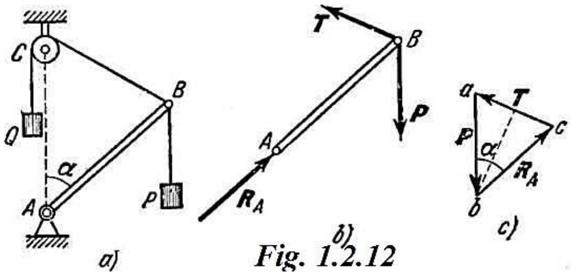


Fig. 1.2.12

Solution. Consider the conditions for the equilibrium of rod AB, to which all the given and required forces are applied. Removing the constraints and treating the rod as a free body (Fig.1.2.12b), draw the forces acting on it: the weight of the load $\boldsymbol{P}$, the tension $\boldsymbol{T}$ in the string, and the reaction $\boldsymbol{R_A}$ of the hinge, which is directed along AB, since in the present case the rod can only be in tension or in compression. If the friction of the rope on the pulley is neglected, the tension in the string can be regarded as uniform throughout its length, whence $\mathrm{T} = \mathrm{Q}$.

For the graphical method of solution, construct a closed force triangle abc with forces $\boldsymbol{P}, \boldsymbol{T}, \boldsymbol{R_A}$ as its sides (Fig.1.2.12c) starting with force $\boldsymbol{P}$**.** As triangles abc andABC are similar, we have $\mathrm{ab} = \mathrm{bc}$ and $\angle \mathrm{abc} = \alpha$. Hence, as $\mathrm{T} = \mathrm{Q} = 2\mathrm{P}\sin\alpha$,

$$R_A = P \text{ and } \sin\frac{\alpha}{2} = \frac{Q}{2P}.$$

It follows from these results that at $< 180°$ equilibrium is possible only if $Q < 2PQ$ and that the rod will be compressed with a force equal to P at any values of Q and α.

Note that force $\boldsymbol{Q}$ (the weight of the load) was not directly included in the equilibrium condition (in the force triangle), as it is applied to the load and not to the rod whose equilibrium was considered.

Problem 6. A crane held in position by a journal bearing A and a thrust bearing B carries a load P (Fig.1.2.13). Neglecting the weight of the structure determine the reactions R_A and R_B of the constraint if the jib is of length l and $AB = h$.

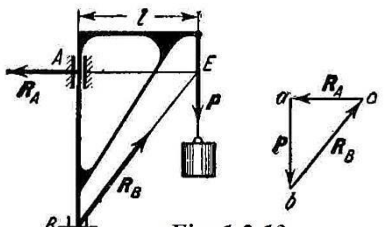


Fig. 1.2.13

Solution. Consider the equilibrium of the crane as a whole under the action of the given and required forces. Mentally remove the constraints and draw the given force $\boldsymbol{P}$ and the reaction $\boldsymbol{R_A}$ of the journal bearing perpendicular to AB. The reaction $\boldsymbol{R_B}$ of the thrust bearing can have any direction in the plane of the diagram. But the crane is in equilibrium under the action of three forces and consequently their lines of action must intersect in one point. This point is E, where the lines of action of $\boldsymbol{P}$ and $\boldsymbol{R_A}$ cross. Hence, the reaction $\boldsymbol{R_B}$ is directed along BE.

To solve the problem by the graphical method draw a close triangle abc with forces $\boldsymbol{P}$, $\boldsymbol{R}_A, \boldsymbol{R}_B$ as its sides starting with the given force $\boldsymbol{P}$. From the similarity of triangles abc and ABE we obtain:

$$\frac{R_A}{P} = \frac{1}{h}, \qquad \frac{R_B}{P} = \frac{\sqrt{h^2 + l^2}}{h},$$

whence

$$R_A = \frac{1}{h} P,\ R_B = \sqrt{1 + \frac{l^2}{h^2}} P.$$

From triangle abc we see that the directions of the reactions $\boldsymbol{R}_A$ and $,\boldsymbol{R}_B$ where draw correctly. The loads acting on the journal bearing A and the thrust bearing B are respectively equal in magnitude to R_A and $,R_B$ but opposite in sense. The greater the ratio $l/_h$ the greater the load acting on the constraints.

This problem is an example of the application of the theorem of three forces. Note the following conclusion arising from it: *if the statement of a problem gives the linear dimensions of structural elements, it is more convenient to solve the force triangle by the rule of similarity; if the angles are given, the formulas of trigonometry should be used.*

Problem 7. A horizontal force $\boldsymbol{P}$ is applied to hinge A of the toggle-press in Fig.1.2.14a. Neglecting the weight of the rods and piston, determine the force exerted by the piston on body M when the given angles are are α and β.

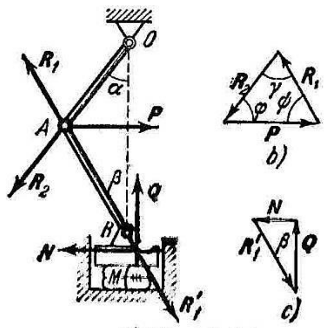


Fig.1.2.14

Solution. First consider the equilibrium of the hinge A to which the given force $\boldsymbol{P}$ is applied. Regarding the hinge as a free body, we find that also acting on it are the reactions $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ of the rods directed along them. Construct a force triangle (Fig. 1.2.14b). Its angles are $\varphi = 90° - \alpha, \psi = 90° - \beta,\ \gamma = \alpha + \beta$. By the law of sines we have:

$$\frac{R_1}{\sin\varphi} = \frac{P}{\sin\gamma},\ R_1 = \frac{P\cos\alpha}{\sin(\alpha+\beta)}.$$

Now consider the equilibrium of the piston, regarding it as a free body. Acting on it are three forces: $\boldsymbol{R}_1'$ exerted by rod AB, the reaction $\boldsymbol{N}$ of the wall, and the reaction $\boldsymbol{Q}$ of the pressed body. The three forces are in equilibrium, consequently they are concurrent. Constructing a triangle with the forces as its sides, (Fig. 1.2.14c), we find:

$$Q = R'_1 \cos\beta.$$

Substituting for $\boldsymbol{R}_1'$ its equivalent $\boldsymbol{R}_1$, we finally obtain:

$$Q = \frac{P\cos\alpha\cos\beta}{\sin(\alpha+\beta)} = \frac{P}{\tan\alpha + \tan\beta}.$$

The force with which the piston compresses the body M is equal to Q in magnitude and opposite in sense.

From the last formula we see that with a constant applied force P the presume Q increases as the angles α and β diminish.

If the rods OA and AB are of equal length, then α=β and $Q = 0.5P\cot\alpha$.

The following conclusion can be drawn from this solution: i*n some problems the given force or forces are applied to one body and the required force or forces act on another; in such cases the equilibrium of the first body is considered and the force with which it acts on the other body is found; then the second body is examined and the required quantities are obtained.*

Problem 8. Rods AB and BC of the bracket in Fig.1.2.15a are joined together and attached to the wall by hinges. Over the pulley at B passes a string one end of which is fastened to the wall while the other supports a load of weight Q. Neglecting the weight of the rods and the diameter of the pulley, determine the reactions of the rods if angles α and β are given.

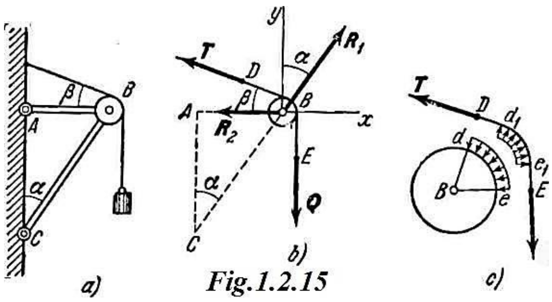


Fig.1.2.15

Solution. Consider the equilibrium of the pulley. Isolate it and draw the reactions of the constraints (Fig.1.2.15b). Acting on the pulley and the segment of the string passing over it are four external forces: the tension $\boldsymbol{Q}$ in the right-hand part of the string, the tension $\boldsymbol{T}$ in the left-hand part of the string, which is equal to Q in magnitude and the reactions $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ of the rods directed along the rods. Neglecting the diameter of the pulley, the forces can be treated as concurrent. As there are more than three forces, the analytical method of solution is more convenient. Draw the coordinate axes as shown in the Fig.1.2.15b and write equations of equilibrium, substituting for T the equal quantity Q:

$$-Q\cos\beta + R_1\sin\alpha - R_2 = 0,$$
$$-Q + Q\sin\beta + R_1\cos\alpha = 0.$$

From the second equation, we find:

$$R_1 = \frac{1-\sin\beta}{\cos\alpha}Q.$$

Substituting this value of R_1 into the first equation and transposing, we obtain:

$$R_2 = \frac{\sin\alpha - \cos(\alpha-\beta)}{\cos\alpha}Q.$$

It should be noted that if, in drawing the reaction of constraints, any reaction is pointed in the wrong direction, this will show up immediately. In analytical method the sign of the respective reaction will be negative.

Problem 9. The vertical pole OA in Fig.1.2.16 is anchored down by guy wires AB and AD which make equal angle $\alpha = 30°$ with the pole; the angle between the planes AOB and AOD is $\varphi = 60°$. Two horizontal wires parallel to the axes Ox and Oy are attached to the pole, and the tension in each of them is $P = 1\ kN$. Neglecting the weight of all the elements, determine the vertical load acting on the pole and the tensions in the guy wires.

Solution. Consider the equilibrium of point A to which the guys and horizontal wires are attached. Acting on it are the reactions $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ of the horizontal wires ($P_1 = P_2 = P$), the reactions R_2 and R_3 of the guys, and the reaction $\boldsymbol{R}_1$ of the pole. The force system is three-dimensional, and the analytical method of solution is most suitable.

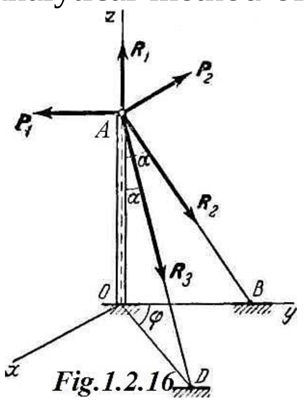


Fig.1.2.16

From the equilibrium equations we have:

$$-P + R_3 \sin\alpha \sin\varphi = 0,$$
$$-P + R_2 \sin\alpha + R_3 \sin\alpha \cos\varphi = 0,$$
$$R_1 - R_2 \cos\alpha - R_3 \cos\alpha = 0,$$

solving which we obtain:

$$R_1 = P\left(1 + \tan\frac{\varphi}{2}\right)\cot\alpha, \qquad R_2 = P\frac{1-\cot\varphi}{\sin\alpha}, R_3 = \frac{P}{\sin\alpha \sin\varphi}.$$

The results show that at $\varphi < 45°$, $R_2 < 0$, and the reaction R_2 is of opposite sense than shown in the diagram. As a wire cannot be in compression, it follows that the guy AD should be anchored in such a way that angle φ would be greater than 45°. Substituting the scalar quantities in the equations, we obtain:

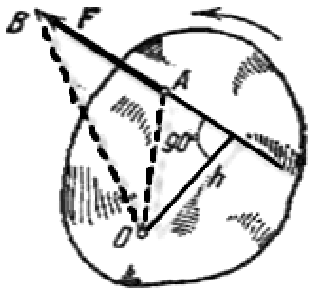
$$R_1 = 2.73\ kN, R_2 0.85\ kN, R_3 = 2.31\ kN.$$

1.3. Parallel Force System

1.3.1. Moment of Force about an Axis or a Point

We know that force acting on a body tends either to displace it in some direction or to rotate it about a point (axis). The tendency of a force to turn a body about a point or an axis is described by the moment of that force.

Consider a force $\boldsymbol{F}$applied at a point A of a rigid body (Fig.1.3.1) which tends to rotate the body about a point O. The perpendicular distance h from O to the line of action of $\boldsymbol{F}$ is called the moment arm of force $\boldsymbol{F}$ about the centre O. It is obvious that the rotational action of any force depends on:



$m_0\,(F) = +Fh$

Fig.1.3.1

1) the magnitude of the force $\boldsymbol{F}$ and the length of its moment arm h;

2) the position of the plane OAB of rotation through the centre O and the force $\boldsymbol{F}$ and

3) the sense of the rotation in that plane.

For the present we shall limit ourselves to coplanar force systems and we needn't define the plane of rotation specially. Thus, we may formulate the concept of moment of a force as a measure of the force tendency to turn the body: *the moment of a force about a center is the product of the force magnitude and the length of the moment arm taken with appropriate sign.*

We shall consider a moment positive if the applied force tends to rotate the body counterclockwise, and negative if it tends to rotate the body clockwise.

We shall denote the moment of a force $\boldsymbol{F}$ about a center O by the symbol $m_o(\boldsymbol{F})$. Thus,

$$m_0(\boldsymbol{F}) = \pm Fh. \tag{1.3.1}$$

It should be noted the following properties of the moment of the force:

-the moment of a force does not change if the point of application of the force is transferred along its line of action;

-the moment of a force about a center can be zero only if the force is zero or if its line of action passes through that center.

-the magnitude of the moment is represented by twice the area of the triangle OAB (Fig. 1.3.1): $m_0(\boldsymbol{F}) = \pm 2$ areas of $\Delta\ OAB$.

It is easy to prove the following theorem (Varignon's theorem): the moment of the resultant of coplanar forces about any center is equal to the algebraic sum of the moments of the component forces about that center.

1.3.2. Composition of Parallel Forces

Let's consider two possible cases: the forces are of the same sense, and the forces are of opposite sense.

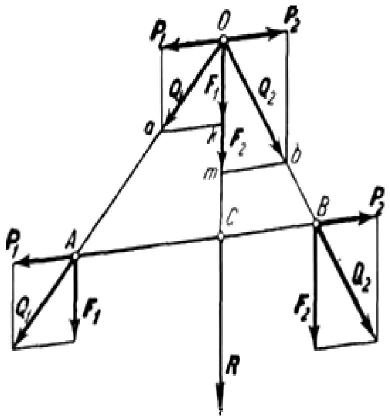


Fig.1.3.2

Consider two parallel forces F_1 and $\boldsymbol{F}_2$. By applying the first and second axioms of statics we can replace the given system of parallel forces with an equivalent system of concurrent forces $\boldsymbol{Q}_1$, and $\boldsymbol{Q}_2$. For this, apply two balanced forces $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ $(\boldsymbol{P}_1 = \boldsymbol{P}_2)$ directed along AB at points A and B, compound them with forces F_1 and $\boldsymbol{F}_2$ according to the parallelogram law, transfer the resultants Q_1 and $\boldsymbol{Q}_2$ to the point O where their lines of action intersect, and resolve them into their initial components. As a result we have obtained two balanced forces $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ applied at point O, which can be neglected, and two forces F_1 and $\boldsymbol{F}_2$ directed along the same line. Now transfer the latter two forces to C and replace them by their resultant $\boldsymbol{R}$ of magnitude $\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2$. Thus, force $\boldsymbol{R}$ is the resultant of the parallel forces F_1 and $\boldsymbol{F}_2$. To determine the position of C consider the triangles OAC, Oak, and OCB, Omb. From the similarity of the respective triangles it follows that

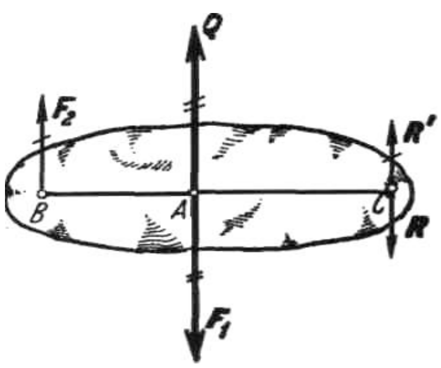


Fig.1.3.3

$$\frac{AC}{OC} = \frac{P_1}{F_1} \text{ and } \frac{BC}{OC} = \frac{P_2}{F_2}. \quad AC \times F_1 = BC \times F_2,$$

as $P_1 = P_2$. From the property of proportion, and taking into account that $BC + AC = AB$ and $\boldsymbol{F}_1 + \boldsymbol{F}_2 = \boldsymbol{R}$, we obtain:

$$\frac{BC}{F_1} = \frac{AC}{F_2} = \frac{AB}{R}. \qquad (1.3.2)$$

Hence, the resultant of two parallel forces of the same sense is equal to the sum of their magnitudes, parallel to them, and is of same sense; the line of action of the resultant is between the points of application of the component forces, its distances from the points being inversely proportional to the magnitudes of the forces.

2) Composition of two forces of opposite sense.

Consider the concrete case of $F_1 > F_2$ (Fig.1.3.2). Take a point C on the extension of BA and apply two balanced forces $\boldsymbol{R}$ and $\boldsymbol{R}'$ parallel to the given forces $\boldsymbol{F}_1$, and $\boldsymbol{F}_2$.

The magnitude of $\boldsymbol{R}$ and $\boldsymbol{R}'$ and the location of C are chosen so as to satisfy the equations:

$$\boldsymbol{R} = \boldsymbol{F}_1 - \boldsymbol{F}_2; \frac{BC}{F_1} = \frac{AC}{F_2} = \frac{AB}{R}.$$

Compounding forces F_2 and $\boldsymbol{R}'$, we find that their resultant $\boldsymbol{Q}$ is equal in magnitude to $\boldsymbol{F}_2 + \boldsymbol{R}'$, i.e., to force $\boldsymbol{F}_1$. It also follows from Eq. (1.3.2) that this force is applied at point A . Forces $\boldsymbol{F}_1$ and $\boldsymbol{Q}$ are balanced and can be discarded. As a result, the given forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are replaced by a single force $\boldsymbol{R}$, their resultant. Thus, the resultant of two parallel forces of opposite sense is equal in magnitude to the difference between their magnitudes, parallel to them, and has the same sense as the greater force; the line of action of the resultant lies on the extension of the line segment connecting the points of application of the component forces, its distances from the points being inversely proportional to the forces.

1.3.3. A Force Couple. Moment of a Couple

A force couple is a system of two parallel forces of same magnitude and opposite sense. A force system constituting a couple is not in equilibrium. This conclusion follows from the first axiom of statics. Furthermore, unlike previously examined systems, a couple has no resultant. For if the couple $(\boldsymbol{F},\ \boldsymbol{F}')$ (Fig.1.3.4) had a resultant, there would also have to be a force $\boldsymbol{Q}_1 = -\boldsymbol{Q}$ capable of balancing it, i.e., the system of forces $\boldsymbol{F}, \boldsymbol{F}^{'}, \boldsymbol{Q}_1$, would be in equilibrium. But as will be shown later on, for any system of forces to be in equilibrium their geometrical sum must be zero. In the present case this would require that $\boldsymbol{F} + \boldsymbol{F}^{'} + \boldsymbol{Q}_1 = 0$ which is impossible since $\boldsymbol{F} + \boldsymbol{F}' = 0$ but $\boldsymbol{Q}_1 = 0$.Thus, a couple cannot be replaced or balanced by a single force.

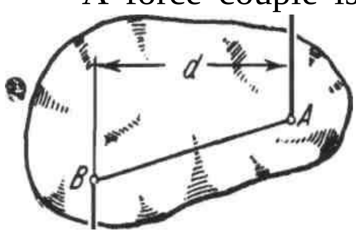


Fig.1.3.4

The plane through the lines of action of both forces of a couple is called the plane of action of the couple. The perpendicular distance d between the lines of action of the forces is called the arm of the couple (Fig.1.3.4). The action of a couple on a rigid body is a tendency to turn it; it depends on the magnitude of the forces of the couple, arm of the couple and the sense of rotation in that plane.

A couple is characterized by its moment. In this section we discuss the properties of couples of coplanar forces. Thus, we can give the following definition: *the moment of a couple is defined as a quantity equal to the product of the magnitude of one of the forces of the couple and the perpendicular distance between the forces, taken with the appropriate sign.* The moment of a couple is said to be positive if the action of the couple tends to turn a body counterclockwise, and negative if clockwise.

Denoting the moment of a couple by the symbol m or M, we have:

$$m = \pm F \times d. \tag{1.3.3}$$

It is apparent that the moment of a couple is equal to the moment of one of its forces about the point of application of the other force i.e.:

$$m = m_B(\boldsymbol{F}) = m_A(\boldsymbol{F'}).$$

Theorem: *the algebraic sum of the moments of the forces of a couple about any point in its plane of action is independent of the location of that point and is equal to the moment of the couple*.

For, taking an arbitrary point O in the plane of a couple (Fig.1.3.5), we find: $m_0(\boldsymbol{F}) = -F \times Oa,\ \ m_0(\boldsymbol{F'}) = F' \times Ob$. Adding the two equations and noting that $F' = F$ and $Ob - Oa = d$, where d is the couple arm, we obtain:

$$m_0(\boldsymbol{F}) + m_0(\boldsymbol{F'}) = m.$$

Let us now formulate two very useful theorems without proving them:

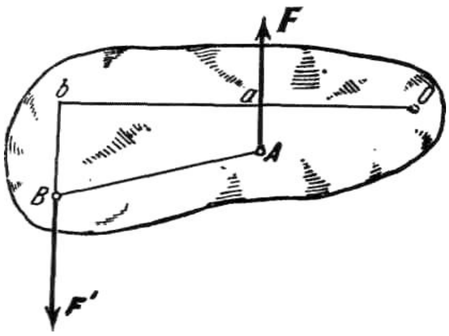


Fig.1.3.5

1) *A couple can be replaced by any other couple of the same moment lying in the same plane without altering the external effect.*

2) *The external effect of a couple on a rigid body remains the same if the couple is transferred from a given plane into any other parallel plane.*

It follows from the first of these theorems two properties:

-a couple can be transferred anywhere in its plane of action;

-it is possible to change the magnitudes of the forces of a couple or its arm arbitrarily without changing its moments.

1.3.4. Composition of Couples. Conditions for the Equilibrium of Couples

Theorem: *a system of coplanar couples is equivalent to a single couple lying in the same plane the moment of which equals the algebraic sum of the moments of the component couples.*

Let three couples of moments m_1, m_2 and m_3 be acting on a body (Fig.1.3.6). By the theorem of equivalent couples they can be replaced by couples $(\boldsymbol{P}_1,\ \boldsymbol{P'}_1)$, $(\boldsymbol{P}_2, \boldsymbol{P'}_2)$, and $(\boldsymbol{P}_3, \boldsymbol{P'}_3)$ with a common arm d and the same moments:

$$P_1 d = m_1,\ \ -P_2 d = m_2, \qquad P_3 d = m_3.$$

Compounding the forces applied at A and B respectively, we obtain a force $\boldsymbol{R'}$ at A and a force $\boldsymbol{R}$ at B magnitudes of which are:

$$R = R' = P_1 - P_2 + P_3.$$

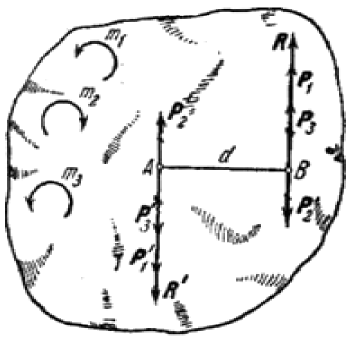


Fig.1.3.6

As a result the set of couples is replaced by a single couple $(\boldsymbol{R}_1, \boldsymbol{R}')$with a moment

$$M = Rd = P_1 d + (-P_2 d) + P_3 d = m_1 + m_2 + m_3.$$

The same results can be obtained for any number of couples, and a set of $n$couples of moments $m_1, m_2 \dots\ m_n$ can be replaced by a single couple with a moment $M = \sum m_k$.

It follows from proved theorem that *for a coplanar system of couples to be in equilibrium it is necessary and sufficient for the algebraic sum of their moments to be zero:*

$$\sum m_k = 0.$$

1.3.5. Illustrative problems

Problem 1. The bracket $ABCD$ in Fig.1.3.7 is in equilibrium under the action of two parallel force $\boldsymbol{P}$ and $\boldsymbol{P'}$ making a couple. Determine the load on the supports if $AB = a = 0.5\ m$, $BC = b = 0.3\ m$, $CD = c = 0.2\ mCD$ and $P = P' = 300\ N$.

Solution. Replace couple $(\boldsymbol{P}, \boldsymbol{P'})$ with an equivalent couple $(\boldsymbol{Q}, \boldsymbol{Q'})$ whose two forces are directed along the reactions of the constraints. The moments of the two couples are equal, i.e., $P(c-a) = Q$, consequently the loads on the constraints are:

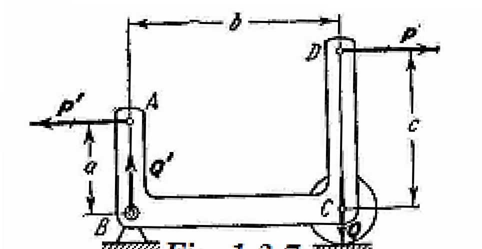


Fig. 1.3.7

$$Q = Q' = \frac{c-a}{b} P = 50\ N$$

and are directed as shown in the diagram.

Problem 2. A couple of moment m_1 acts on gear 1 of radius r_1 in Fig.1.3.8a. Determine the moment m_2 of the couple which should be applied to gear 2 of radius $r_2$in order to keep the system in equilibrium.

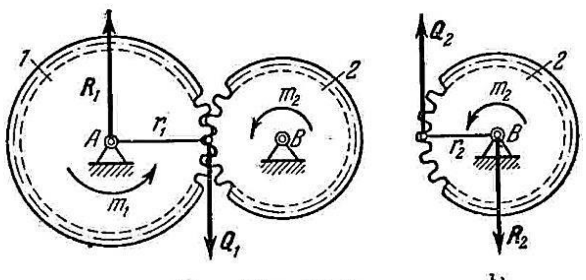


Fig. 1.3.8

Solution. Consider first the conditions for the equilibrium of gear 1. Acting on it is the couple of moment m_1 which can be balanced only by the action of another couple, in this case the couple $(\boldsymbol{Q_1}, \boldsymbol{R_1})$ created by the force $\boldsymbol{Q_1}$ exerted on the tooth of gear 1 by gear 2, and the reaction $\boldsymbol{R_1}$ of the axle A. So we have $m_1 - Q_1 r_1$ or $Q_1 = {}^{m_1}/_{r_1}$.

Consider now the condition for the equilibrium of gear 2. By the fourth principle we know that gear 1 acts on gear 2 with a force $\boldsymbol{Q_2} = -\boldsymbol{Q_1}$ (Fig.1.3.8b), which together with the reaction of axle B makes a couple $(\boldsymbol{Q_2}, \boldsymbol{R_2})$ of moment –

Q_2r_2. This couple must be balanced by a couple of moment m_2 acting on gear 2. Thus, $m_2 + Q_2r_2 = 0$. Hence, as $Q_2 = Q_1$,

$$m_2 = \frac{r_2}{r_1} m_1.$$

It will be noticed that the couples of moments m_1 and m_2 do not satisfy the equilibrium condition, which could be expected, as the couples are applied to different bodies.

The force $\boldsymbol{Q_1}$ (or $\boldsymbol{Q_2}$) obtained in the course of the solution is called the circumferential force acting on the gear. The circumferential force is thus equal to the moment of the acting couple divided by the radius of the gear:

$$Q_1 = \frac{m_1}{r_1} = \frac{m_2}{r_2}$$

Problem. 3. The travelling crane in Fig.1.3.9 weighs $P = 4\ t$, its centre of gravity lies on DE, it lifts a load of weight $Q = 1\ t$, the length of the jib (the distance of the load from DE) is $b = 3.5\ m$, and the distance between the wheels is $AB = 2a = 2.5\ m$. Determine the force with which the wheels A and B act on the rails.

Solution. Consider the equilibrium of the crane-and-load system taken as a free body: the active forces are $\boldsymbol{P}$ and $\boldsymbol{Q}$, the unknown forces are the reactions $\boldsymbol{N}_A$ and $\boldsymbol{N}_B$ of the removed constraints. Taking A as the centre of the moments of all the forces and projecting the parallel forces on a vertical axis, we obtain

$$-Pa + N_B \cdot 2a - Q(a+b) = 0, N_A + N_B - P - Q = 0,$$

solving which we find:

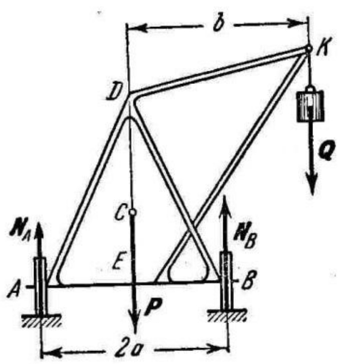


Fig. 1.3.9

$$N_A = \frac{P}{2} - \frac{Q}{2}\left(\frac{b}{a} - 1\right) - 1.1\ t;$$

$$N_B = \frac{P}{2} + \frac{Q}{2}\left(\frac{b}{a} + 1\right) = 3.9\ t.$$

To verify the solution, write the equation of the moments about B:

$$-N_A 2a + Pa - Q(b-a) = 0.$$

Substituting the value of N_A, we find that the equation is satisfied. The pressures exerted by the wheels on the rails are respectively equal to N_A and N_B in magnitude and directed vertically down.

From the solution we see that at

$$Q = \frac{a}{b-a} P = 2.22\ t$$

the reaction $\boldsymbol{N_A}$ is zero and the left wheel no longer pressed on the rail. If the load Q is further increased the crane will topple over. The maximum load Q at which equilibrium is maintained is determined by the equation $\sum m_B(\boldsymbol{F}_k) = 0$.

1.4. COPLANAR FORCE SYSTEM

1.4.1. Reduction of a Coplanar Force System to a given Centre

Theorem: *Force acting on a rigid body can be moved parallel to its line of action to any point of the body, if we add a couple of a moment equal to the moment of the force about the point to which it is translated.*

Consider a force $\boldsymbol{F}$ applied to a rigid body at a point A (Fig.1.4.1a). The action of the force will not change if two balanced forces $\boldsymbol{F}' = \boldsymbol{F}$ and $\boldsymbol{F}" = -\boldsymbol{F}$ are applied at any point B of the body. The resulting three-force system consists of a force $\boldsymbol{F}'$ equal to $\boldsymbol{F}$ and applied at B, and a couple $(\boldsymbol{F},\ \boldsymbol{F}")$ of moment $m = m_B(\boldsymbol{F})$.

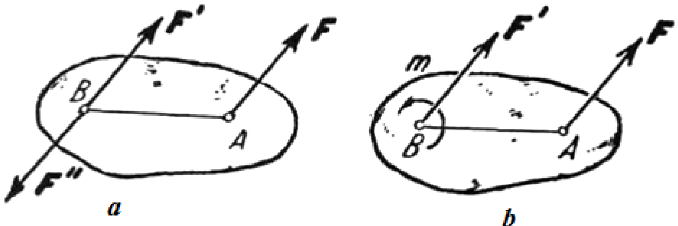


Fig.1.4.1

The result can also be denoted as in Fig.1.4.1b, with force $\boldsymbol{F}$ neglected.

Let a set of coplanar forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$,…$\boldsymbol{F}_n$ be acting on a rigid body and let O be any point coplanar with them which we shall call the center of reduction. By the theorem just proved, we can transfer all the forces to O as in Fig. 1.4.2a. As a result we have a system of forces $\boldsymbol{F'}_1 = \boldsymbol{F}_1, \boldsymbol{F'}_2 = \boldsymbol{F}', \dots, \boldsymbol{F'}_n = \boldsymbol{F}_n$, and couples of moments $m_1 = m_0(\boldsymbol{F}_1), m_2 = m_0(\boldsymbol{F}_2), \dots\ m_n = m_0(\boldsymbol{F}_n)$.

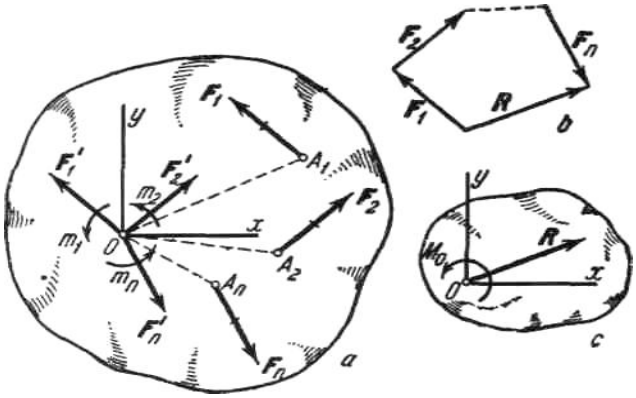


Fig 1.4.2

The forces applied at O can be replaced with their resultant $\boldsymbol{R}' = \sum_k \boldsymbol{F'}_k$, acting at that point O , or

$$\boldsymbol{R} = \sum_k \boldsymbol{F}_k. \tag{1.4.1}$$

The quantity $\boldsymbol{R}$, which is the geometrical sum of all the forces of the given system is called the principal vector of the system.

Similarly, by the theorem of composition of couples, we can replace all the couples with a coplanar resultant with a moment:

$$M_0 = \sum_k m_k = \sum_k m_0\,(\boldsymbol{F}_k). \tag{1.4.2}$$

The quantity M_0 which is the sum of the moments of all the forces of the system about centre O is called the principal moment of the force system about centre$O$.

Thus, any system of coplanar forces can be reduced to an arbitrary center in such a way that it is replaced by a single force equal to the principal vector of the system and applied at the center of reduction and a single couple of moment equal to the principal moment of the system about center of reduction (Fig.1.4.2c).

It should be noted that $\boldsymbol{R}$ is not the resultant of the force system, as it replaces the system only together with a couple.

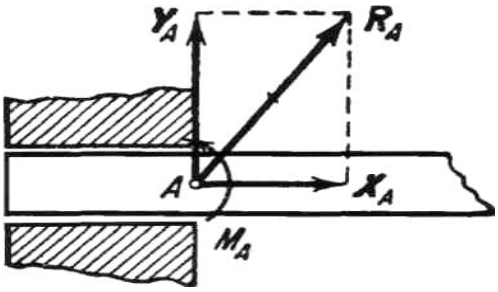


Fig.1.4.3

Hence, in order to define a coplanar force system it is sufficient to define its principal vector and principal moment about any center of reduction. It is apparent that two force systems with equal principal vectors and principal moments are statically equivalent.

Let us illustrate the results just obtained considering the reaction of fixed support (rigid clamp or embedding) (Fig.1.4.3). In this case the action of the constraining surfaces on the embedded portion of the beam is that of a system of distributed forces of reaction.

By reducing the forces of reaction to a common center A, we can replace them with an immediately unknown force attached at A and a couple M_A of immediately unknown moment. Force $\boldsymbol{R}$ can in turn be denoted by its rectangular components $\boldsymbol{X}_A$ and $\boldsymbol{Y}_A$.

Thus, to determine the reactions of a rigid clamp we must find three unknown quantities X_A, Y_A, and M_A.

1.4.2. Reduction of a Coplanar Force System to the Simplest Possible Form

The results obtained in section 1.4.1. make it possible to reduce a given coplanar force system to the simplest possible form. Let us consider the following cases:

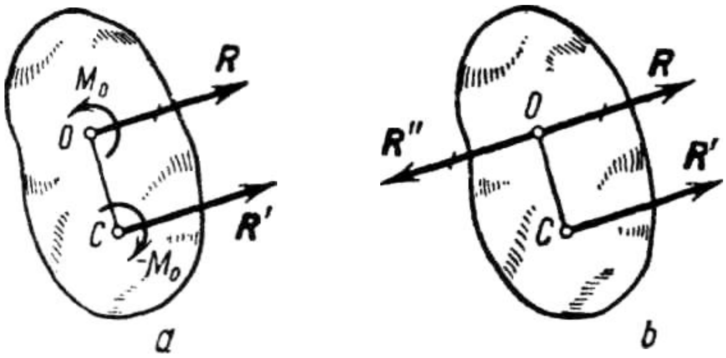


Fig.1.4.4

1) If $\boldsymbol{R} = 0$ and $M_0 = 0$ the system is in equilibrium. This case will be examined in the following section.

2) If $\boldsymbol{R} = 0$ and $M_0 \neq 0$, the system can be reduced to a couple of moment $M_0 = \sum m_0(\boldsymbol{F}_k)$ equal to the principal moment of the system. In this case the magnitude of M_0 does not depend on the location of the centre O, otherwise we would find that the same system could be replaced by non-equivalent couples, but this is impossible.

3) If $\boldsymbol{R} \neq 0$ and $M_0 = 0$. In this case the system can immediately be replaced by a single force, i.e., the resultant $\boldsymbol{R}$ which passes through centre O.

4) If $\boldsymbol{R} \neq 0$ and $M_0 \neq 0$ (Fig.1.4.4a). In this case the couple of moment M_0 can be represented by two forces $\boldsymbol{R}'$and $\boldsymbol{R}''$, such that $\boldsymbol{R}' = \boldsymbol{R}$, and $\boldsymbol{R}'' = \boldsymbol{R}$ (Fig.1.4.4b). The arm of the couple ($\boldsymbol{R}', \boldsymbol{R}''$) must be $d = {}^{|M_0|}/_{R}$.

Discarding the mutually balanced forces $\boldsymbol{R}$ and $\boldsymbol{R}''$ we find that the whole force system can be replaced by the resultant $\boldsymbol{R}' = \boldsymbol{R}$ passing through point *C*.

These cases show that if a coplanar force system is not in equilibrium, it can be reduced either to a resultant (when $\boldsymbol{R} \neq 0$) or to a couple (when $\boldsymbol{R} = 0$).

1.4.3. Conditions for the Equilibrium of a Coplanar Force System

For any given coplanar force system to be in equilibrium it is necessary and sufficient for the following two conditions to be satisfied simultaneously:

$$\boldsymbol{R} = 0,\ M_0 = 0,$$

where O is any point in a given plane, as at $\boldsymbol{R} = 0$ the magnitude of M_0 does not depend on the location of O.

These conditions are necessary, for if one of them is not satisfied the force system is reduced either to a resultant (when $\boldsymbol{R} \neq 0$) or to a couple (when $M_0 \neq 0$) and consequently is not balanced. At the same time, these conditions are sufficient, for at $\boldsymbol{R} \neq 0$ the system can be reduced only to a couple of moment M_0, but $M_0 = 0$, hence the system is in equilibrium.

Let us consider three different forms of the analytical conditions of equilibrium.

1. *The basic form of the equations.* The magnitude of $\boldsymbol{R}$ and M_0 are determined by the equations :

$$R = \sqrt{R_x^2 + R_y^2},\quad M_0 = \sum_k m_0\,(\boldsymbol{F}_k).$$

where $R_x = \sum_k F_{kx}$. and $R_y = \sum_k F_{ky}$. But R can be zero only if both $R_x = 0$ and $R_y = 0$. Hence, conditions of equilibrium will be satisfied if

$$\sum_k F_{kx} = 0,\ \sum_k F_{ky} = 0,\ \sum_k m_0\,(\boldsymbol{F}_k) = 0. \qquad (1.4.3)$$

Eqs. (1.4.3) express the following analytical conditions of equilibrium: *for any given coplanar force system to be in equilibrium it is necessary and sufficient for the sums of the projections of all the forces on each of the two coordinate axes and the sum of the moments of all the forces about any point in the plane to be separately zero.*

In the mechanical sense the first two equations express the necessary conditions for a body to have no translation parallel to the coordinate axes, and the third equation expresses conditions for it to have no rotation in the plane Oxy.

2. *The second form of the equations. For any given coplanar force system to be in equilibrium it is necessary and sufficient for the sums of the moments of all the forces about any two points A and B , and the sum of the projections of all the forces on any axis OX not perpendicular to AB to be separately zero:*

$$\sum m_A\,(\boldsymbol{F}_k) = 0,\ \ \sum m_B\,(\boldsymbol{F}_k) = 0,\ \ \sum F_{kx} = 0. \qquad (1.4.4)$$

The necessity of these conditions is apparent, for if any one of them is not satisfied, then either $\boldsymbol{R} \neq 0$ or $M_A \neq 0$ $(M_B \neq 0)$ and the forces will not be in equilibrium.

These conditions are sufficient, for if for a given force system only the first two of Eqs. (1.4.4) are satisfied, then $M_A = 0$ and $M_B = 0$. By p.1.4.2, such a force system may not be in equilibrium as it may have a resultant $\boldsymbol{R}$ passing through the points A and B (Fig.1.4.5). But from the third equation we must have $R_x = \sum F_{kx} = 0$. As OX is not perpendicular to AB, the latter condition can be satisfied only if the resultant $\boldsymbol{R}$ is zero, i.e., the system is in equilibrium.

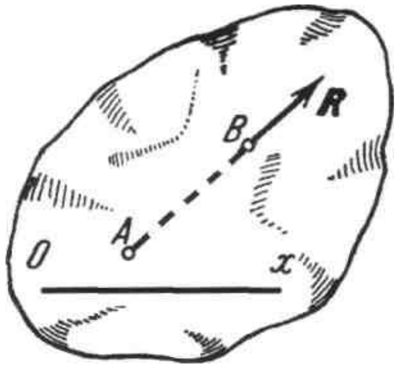


Fig.1.4.5

3. *The third form of the equations. For any given coplanar force system to be in equilibrium it is necessary and sufficient for the sums of the moment of all the forces about any three non-collinear points to be* zero:

$$\sum m_A(\boldsymbol{F}_k) = 0, \quad \sum m_B(\boldsymbol{F}_k) = 0, \quad \sum F_{kx} = 0. \tag{1.4.5}$$

The necessity of these conditions is obvious. Their sufficiency follows from the consideration that if, with all the three equations satisfied, the system would not be in equilibrium, it could be reduced to a single resultant passing through points A, B and C, which is impossible as they are not collinear. Hence, if Eqs. (1.4.5) are satisfied, the system is in equilibrium.

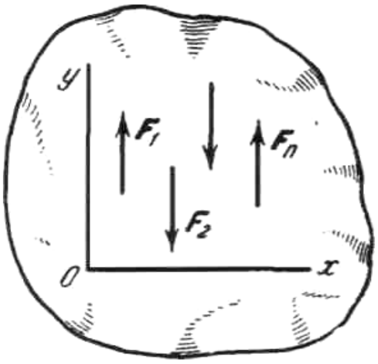


Fig.1.4.6

In all the cases we have three conditions. The Eqs. (1.4.3) are basic because they impose no restrictions on the choice of the coordinate axes or the centers of moments.

Finally, let us consider the particular case, i.e., equilibrium of a coplanar system of parallel forces.

If all the forces acting on a body are parallel (Fig.1.4.6), we can take axis x of a coordinate system perpendicular to them and axis y parallel to them. Then the x projections of all the forces will be zero, and the first one of Eqs. (1.4.3) becomes an identity $0 \equiv 0$. Hence, for parallel forces we have two equations of equilibrium:

$$\sum F_{kx} = 0, \quad \sum m_0(\boldsymbol{F}_k) = 0,$$

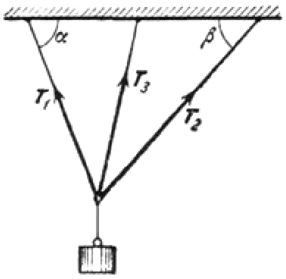
where the y axis is parallel to the forces. Another form of the conditions for the equilibrium of parallel forces follows from Eqs. (1.4.4):

$\sum_k m_A(\boldsymbol{F}_k) = 0$, $\sum_k m_B(\boldsymbol{F}_k) = 0$, Where the points A and B should not lie on a straight line parallel to the given forces.

1.4.4. Statically Determinate and Statically Indeterminate Problems

In problems where the equilibrium of constrained rigid bodies is considered, the reactions of the constraints are unknown quantities. Their number depends on the number and the type of the constraints. A problem of statics can be solved only if the number of unknown reactions is not greater than the number of equilibrium equations which they are present. Such problems are called statically determinate, and the corresponding systems of bodies are called statically determinate systems.

Problems in which the number of unknown reactions of the constraints is greater than the number of equilibrium equations in which they are present are called statically indeterminate, and the corresponding systems of bodies are called statically indeterminate.



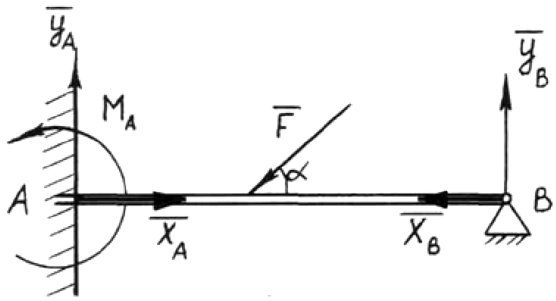


Fig.1.4.7 ***Fig.1.4.8***

The examples of statically indeterminate systems are shown in Fig.1.4.7.

There are three unknown quantities in the problem shown in Fig.1.4.7: the tensions $\boldsymbol{T}_1, \boldsymbol{T}_2$ and $\boldsymbol{T}_3$ of the strings, but only the two equations for the equilibrium of a coplanar system of concurrent forces.

There are five unknown quantities in the problem shown in Fig.1.4.8: X_A, X_B, Y_A, Y_B and M_A, but only the three equations for the equilibrium of a coplanar force system .

The difference between number of unknown quantities and number of equilibrium equations is called degree of the static indeterminateness. In the first case it is equal to one and in the second one it is equal to two. It can be seen that the static indeterminateness of a problem is a result of the presence of too many constraints. In the first problem two strings are sufficient to keep the weight in equilibrium; in the second one the embedding is sufficient to keep the beam in equilibrium.

We shall be concerned only with statically determinate problems. Statically indeterminate problems are solved in the courses of strength of materials and statics of structures.

1.4.5. Equilibrium of Systems of Bodies

In many cases the static solution of engineering structures is reduced to an investigation of systems of connected bodies. We shall call the constraints connecting the parts of a given structure internal, as opposed to external constraints which connect a given structure with other bodies.

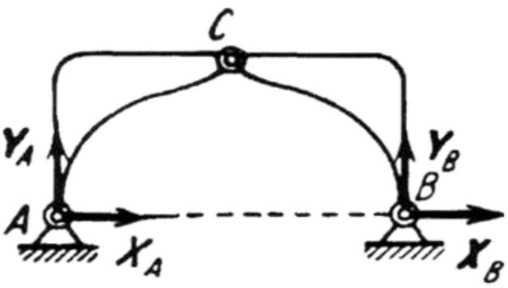


Fig.1.4.9

If a structure remains rigid after the external constraints are removed, the problems of statics are solved for it as for a rigid body. However, engineering structures may not necessarily remain rigid when the external constraints are removed. An example of such a structure is the three-pin arch in Fig.1.4.9.

If supports A and B are removed the arch is no longer rigid for its parts can turn about pin C.

According to the principle of solidification, for a system of forces acting on such a structure to be in equilibrium it must satisfy the conditions of equilibrium for a rigid body. It was pointed out, though, that these conditions, while necessary, were not sufficient, and therefore not all the unknown quantities could be determined from them. In order to solve such a problem it is necessary to examine additionally the equilibrium of one or several parts of the given structure.

One of the ways of solving such a problem is to divide a structure into separate bodies and write the equilibrium equations for each as for a free body (Fig.1.4.10).

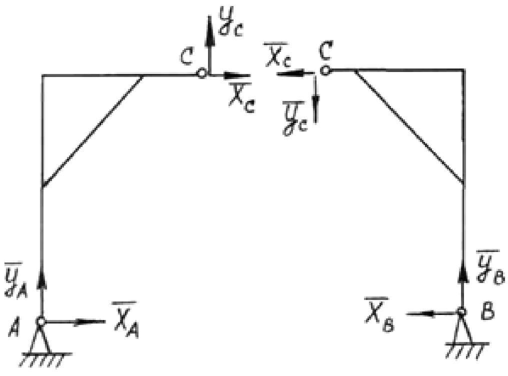


Fig.1.4.10

The reactions of the internal constraints $\boldsymbol{X}_C$ and $\boldsymbol{Y}_C$ will constitute pairs of forces equal in magnitude and opposite in sense in accordance with the fourth axiom of statics. Hence we have six unknown reactions: $X_A, Y_A, X_B, Y_B, X_C, Y_C$.

At the same time we can compose six equations of equilibrium: three equations for the left-hand part of structure, and three ones for the right-hand part. Solving the system of six equations we can determine all six unknown quantities.

For a structure of n bodies, each of which is subjected to the action of a coplanar system, we thus have $3n$ equations from which we may determine unknown quantities.

Another method of solving such problems is to compose three equations of equilibrium for the three-pin arch as a whole, and three more for left or right-hand part of the arch. Then, we shall also have six equations for six unknown reactions.

If the number of unknown quantities is greater than the number of equations, the problem is statically indeterminate.

1.4.6. Illustrative Problems

Problem. 1 One end of a uniform beam AB weighing P N (Fig.1.4.11) rests at A against a corner formed by a smooth horizontal surface and block D, and at B on a smooth plane inclined α degree to the horizontal. The beam's inclination to the horizontal is equal to β. Determine the pressure of the beam on its three constraints.

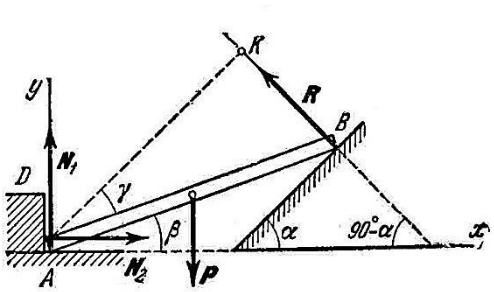


Fig.1.4.11

Solution. Consider the equilibrium of the beam as a free body. Acting on it are the given force $\boldsymbol{P}$ applied at the middle of the beam and the reactions $\boldsymbol{R}$, $\boldsymbol{N}_1$, and $\boldsymbol{N}_2$ of the constraints directed normal to the respective surfaces. Draw the coordinate axes as in Fig.1.4.11 and write the equilibrium equations, taking the moments about A, where two of the unknown forces intersect. First compute the projections of all the forces on the coordinate axes and their moments about A.

The equilibrium equations:

$$N_2 - R\sin\alpha = 0,$$
$$N_1 - P + R\cos\alpha = 0,$$
$$-Pa\cos\beta + 2Ra\cos\gamma = 0.$$

From the last equation we find:

$$R = \frac{P\cos\beta}{2\cos\gamma}.$$

As AK is parallel to the inclined plane, $\angle KAx = \alpha$, whence $\gamma = \alpha - \beta$ and finally

$$R = \frac{P\cos\beta}{2\cos(\alpha-\beta)}.$$

Solving the first two equations, we obtain:

$$N_1 = P\left[1 - \frac{\cos\alpha\cos\beta}{2\cos(\alpha-\beta)}\right], N_2 = P\frac{\sin\alpha\cos\beta}{2\cos(\alpha-\beta)}.$$

The forces exerted on the surfaces are equal in magnitude to the respective reactions and opposite in sense.

The values of N_1 and N_2 can be verified by solving equations of moments about the points of intersection of $\boldsymbol{R}$ and $\boldsymbol{N}_2$ and $\boldsymbol{R}$ and $\boldsymbol{N}_1$.

From this solution we can draw the following conclusion: If, in order to determine the projections or moments of any force or forces, we need to know a quantity (e.g., the length of a line or size of an angle) not given in the statement of the problem, we should denote that quantity by a symbol and include it in the equilibrium equations. If the introduced quantity is not cancelled out in the course of the computations, it should be expressed in terms of given quantities.

Problem. 2 Acting on a symmetrical arch of weight $P = 8\ t$ (Fig.1.4.12) is a set of forces reduced to a force $Q = 4\ t$ applied at D and a couple of moment $M_D = 12\ t \cdot m$. The dimensions of the arch are $a = 10\ m,\ b = 2\ m,\ h = 3\ m,$ and $\alpha = 60°$. Determine the reactions of the pin B and the roller A.

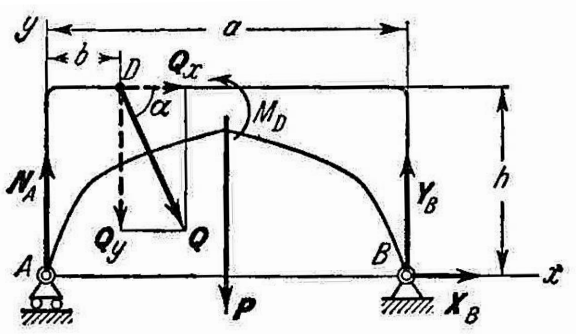


Fig. 1.4.12

Solution. Consider the equilibrium of the arch as a free body. Acting on it are the given forces $\boldsymbol{P}$ and $\boldsymbol{Q}$ and a couple of moment M_D, and also the reactions of the supports $\boldsymbol{N}_A, \boldsymbol{X}_B, \boldsymbol{Y}_B$. In this problem it is more convenient to take the moments about A and B and the force projections on axis Ax. Then each equation will contain one unknown force. In computing the moments of force $\boldsymbol{Q}$, resolve it into rectangular components $\boldsymbol{Q}_x$ and $\boldsymbol{Q}_y$.

Writing the equilibrium equations and taking into account that $|Q_x| = Q\cos\alpha$, and $|Q_y| = Q\sin\alpha$, we obtain:

$$X_B + Q\cos\alpha = 0,$$
$$Y_B a - P\frac{a}{2} - hQ\cos\alpha - bQ\sin\alpha + M_D = 0,$$
$$-N_A a + P\frac{a}{2} - hQ\cos\alpha + (a-b)Q\sin\alpha + M_D = 0.$$

Solving the equations, we find

$$X_B = -Q\cos\alpha = -2t,$$
$$Y_B = \frac{P}{2} + Q\frac{b\sin\alpha + h\cos\alpha}{a} - \frac{M_D}{a} \approx 4.09t,$$
$$N_A = \frac{P}{2} + Q\frac{(a-b)\sin\alpha - h\cos\alpha}{a} + \frac{M_D}{a} \approx 7.37t.$$

The value of X_B is negative, which means that the sense of the x component of the reaction at B is opposite to that shown in the diagram, which could have been foreseen. The total reaction at B can be found from the geometrical sum of the rectangular components $\boldsymbol{X_B}$ and $\boldsymbol{Y_B}$, its magnitude being

$$R_B = \sqrt{X_B^2 + Y_B^2} \approx 4.55t.$$

If the sense of the couple acting on the arch were opposite to that indicated in Fig.1.4.12, we would have $M_D = -12\ t \cdot m$. In this case $Y_B = 6.49\ t, N_A = 4.97\ t, Y_B$, while X_B would remain the same.

To check the solution, write the equation for the projections on axis Ax:

$$N_A + Y_B - P - Q\sin\alpha = 0.$$

Substituting the obtained values of N_A and Y_B, we find that they satisfy the equation (substitution should be carried out in both the general form, to verify the equations, and in the numerical solution to verify the computations).

Problem. 3 The beam in Fig.1.4.13a is embedded in a wall at an angle $\alpha = 60°$ to it. The length of the portion AB is $b = 0.8\, m$ and its weight is $P = 1000\, N$. The beam supports a cylinder of weight $Q = 1800\, N$. The distance AE along the beam from the wall to the point of contact with the cylinder is $a = 0.3\, m$. Determine the reactions of the embedding.

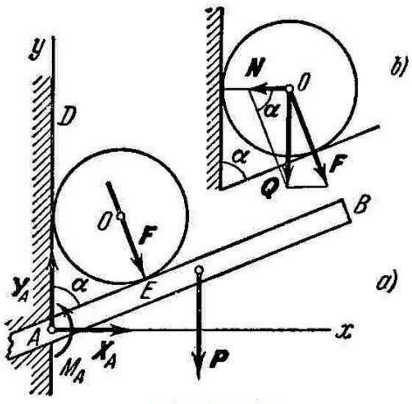


Fig. 1.4.13

Solution. Consider the equilibrium of the beam as a free body. Acting on it are force $\boldsymbol{P}$ applied halfway between A and B, force $\boldsymbol{F}$ applied perpendicular to the beam at E (but not force $\boldsymbol{Q}$, which is applied to the cylinder, no to the beam!), and the reactions of the embedding, indicated by the rectangular components $\boldsymbol{X_A}$ and $\boldsymbol{Y_A}$ and a couple of moment $\boldsymbol{M_A}$.

To determine $\boldsymbol{F}$ we resolve force $\boldsymbol{Q}$, which is applied at the centre of the cylinder, into components $\boldsymbol{F}$ and $\boldsymbol{N}$ respectively perpendicular to the beam and the wall (Fig.1.4.13b). From the parallelogram we obtain:

$$F = \frac{Q}{\sin\alpha}.$$

Writing the equations of equilibrium and substituting the value of F, we have:

$$X_A + Q\cot\alpha = 0,\ \ Y_A - Q - P = 0,\ \ M_A - Q\frac{a}{\sin\alpha} - P\frac{b}{2}\sin\alpha = 0.$$

Solving these equations we find:

$$X_A = -Q\cot\alpha = -1038\, N,$$
$$Y_A = P + Q = 2800\, N,$$
$$M_A = Q\frac{a}{\sin\alpha} + P\frac{b}{2}\sin\alpha = 969\, N\cdot m.$$

The reaction of the wall consists of force $R_A = \sqrt{X_A^2 + Y_A^2}$ and a couple of moment M_A.

The solution of this problem once again underlines the fundamental point: *the equilibrium equations include only the forces acting directly on the body whose equilibrium is being considered.*

Problem 4. A string supporting a weight $Q = 2400\, N$ passes over two pulleys C and D as shown in Fig.1.4.14. The other end of the string is secured at B, and the frame is kept in equilibrium by a guy wire EE_1. Neglecting the weight of the frame and friction in the pulleys, determine the tension in the guy wire and the reactions at A, if the constraint at A is a smooth pivot allowing the frame to turn about its axis. The dimensions are as shown in the diagram.

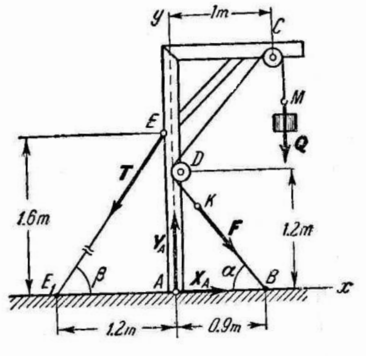


Fig. 1.4.14

Soluton. Consider the whole system of the frame and the portion $KDCM$ of the string as a single free rigid body. Acting on it are the following external forces: the weight $\boldsymbol{Q}$ of the load, the tension $\boldsymbol{F}$ in section DB of the string, and the reactions $\boldsymbol{T}, \boldsymbol{X}_A$, and $\boldsymbol{Y}_A$ of the constraints. The internal forces cancel each other and are not shown in the diagram. As the friction of the pulleys is neglected, the tension in the cable is uniform throughout its whole length and $F = Q$.

Introducing angles α and β, let us compute the projections of all the forces on the coordinate axes and their moments about A.

From the right-angle triangles AEE_1 and ADB we find that $EE_1 = 2.0\, m$ and $DB = 1.5\, m$, whence $\sin\alpha = \sin\beta = 0.8$, $\cos\alpha = \cos\beta = 0.6$, and $\alpha = \beta$. Substituting for the trigonometric functions their values and assuming $F = Q$, the equations of equilibrium give:

$$0.6Q - 0.6T + X_A = 0,$$
$$-Q - 0.8Q - 0.8T + Y_A = 0,$$
$$-1.0Q - 0.72Q + 0.96T = 0,$$

solving which, we find:

$$T = \frac{43}{24}Q = 4300\, N,\;\; X_A = \frac{19}{40}Q = 1140\, N,\;\; Y_A = \frac{97}{30}Q = 7760\, N.$$

Attention is drawn to the following conclusions: 1) in writing equations of equilibrium, any system of bodies which remains fixed when the constraints are removed can be regarded as a rigid body; 2) the internal forces acting on the parts of a system (in the case the tension of the string DC acting on pulleys C and D) are not included in the equilibrium equations as they cancel each other.

*Problem 5.*The horizontal member AD of the bracket in Fig.1.4.15 weighs $P_1 = 150\, N$, and the inclined member CB weighs $P_2 = 120\, N$. Suspended from the horizontal member at D is a load of weight $Q = 300\, N$. Both members are attached to the wall and to each other by smooth pins (the dimensions are shown in the diagram). Determine the reactions at A and C.

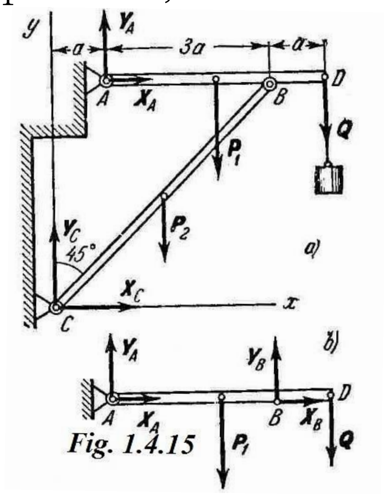


Fig. 1.4.15

Solution. Considering the bracket as a whole as a free body, we find the acting on it are the given forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{Q}$ and the reactions of the supports $\boldsymbol{X}_A, \boldsymbol{Y}_A, \boldsymbol{X}_C, \boldsymbol{Y}_C$. But with its constraints removed the bracket is no longer rigid body, because the members can turn about pin B. On the other hand, by the principle of solidification, if it is in equilibrium the forces acting on it must satisfy the conditions of static equilibrium. We may therefore write the corresponding equations:

$$\sum F_{kx} = X_A + X_C = 0,$$
$$\sum F_{ky} = Y_A + Y_C - P_1 - P_2 - Q = 0,$$
$$\sum m_A(\boldsymbol{F}_k) = X_C 4a - Y_C a - P_2 a - P_1 2a - Q4a = 0.$$

We find that the three equations contain four unknown quantities $\boldsymbol{X}_A, \boldsymbol{Y}_A, \boldsymbol{X}_C, \boldsymbol{Y}_C$. Let us therefore investigate additionally the equilibrium conditions of member AD (Fig.1.4.15b). Acting on it are forces $\boldsymbol{P}_1$, $\boldsymbol{Q}$ and the reactions $\boldsymbol{X}_A, \boldsymbol{Y}_A, \boldsymbol{X}_B, \boldsymbol{Y}_B$. If we write the required fourth equation for the moments of these forces about B we shall avoid introducting two more unknown quantities, $\boldsymbol{X}_B$ and $\boldsymbol{Y}_B$. We have

$$\sum m_B(\boldsymbol{F}_k) = -Y_A 3a + P_1 a - Qa = 0.$$

Solving the system of four equations (starting with the last one) we find:

$$Y_A = \frac{1}{3}(P_1 - Q) = -50\ N,$$
$$Y_C = \frac{2}{3}P_1 + P_2 + \frac{4}{3}Q = 620\ N$$
$$X_C = \frac{2}{3}P_1 + \frac{1}{2}P_2 + \frac{4}{3}Q = 560\ N$$
$$X_A = -X_C = -560\ N.$$

Problem 6. A horizontal force $\boldsymbol{F}$ acts on the three-pin arch in Fig.1.4.16. Show that in determining the reactions of supports A and B force $\boldsymbol{F}$ cannot be transferred along its action line to E.

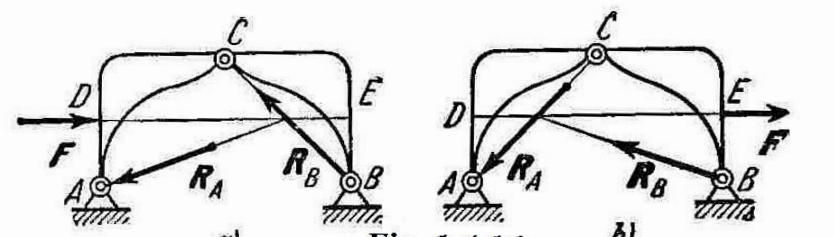


Fig. 1.4.16

Solution. Isolating the arch from its external supports A and B, we obtain a deformable structure which cannot be treated as rigid. Consequently, the point of action of the force acting on the structure cannot be transferred along DE even to determinate the conditions for the equilibrium of the structure.

Let us demonstrate this by solving the problem (the weight of the arch is neglected). Consider first the right-hand member of the arch as a free body. Acting on it are only two forces, the reactions $\boldsymbol{R}_B$ and $\boldsymbol{R}_C$ of the pins B and C. To be in equilibrium, these two forces must be directed along the same line, i.e., along BC, and consequently the reaction $\boldsymbol{R}_B$ is directed along BC.

Investigating now the equilibrium of the arch as a whole, we find that acting on it are three forces, the given force $\boldsymbol{F}$ and the reactions of the supports $\boldsymbol{R}_B$ (whose direction we have established) and $\boldsymbol{R}_A$. From the theorem of three forces we know that if the system is in equilibrium the forces must be concurrent. Thus we obtain the direction of $\boldsymbol{R}_A$. The magnitudes of $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$ can be found by the triangle rule.

If we apply force $\boldsymbol{F}$ at E and, reasoning in the same way, make the necessary constructions (Fig.1.4.16 b), we shall find that the reactions of the supports $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$ are different both in magnitude and in direction.

1.5. FRICTION

1.5.1. Sliding Friction

It is well known from experience that when two bodies tend to slide on each other, a resisting force appears at their surface of contact which opposes their relative motion. This force is called sliding friction.
Friction is due primarily to minute irregularities on the contacting surfaces, which resist their relative motion, and to forces of adhesion between contacting surfaces.
There are several general laws which reflect the principal features of friction. The laws of sliding friction can be formulated as follows:
1.When two bodies tend to slide on each other, a frictional force is developed at the surface of contact, the magnitude of which can have any value from zero to a maximum value which is called limiting friction.
2. Limiting friction is equal in magnitude to the product of the coefficient of static friction (or friction of rest) and the normal pressure or normal reaction:

$$F_l = fN. \tag{1.5.1}$$

The coefficient of static friction f is a dimensionless quantity which is determined experimentally and depends on the material of the contacting bodies and the conditions of their surfaces.
3. Within broad limits, the value of limiting friction does not depend on the area of the surface of contact.
Taken together, the first and second laws state that for conditions of equilibrium the static friction $F \le F_l$.
The coefficient of friction can be determined experimentally by means of a simple device shown in Fig. 1.5.1. The horizontal plate AB and rectangular block D are made of materials for which the coefficient of friction is to be determined. By gradually loading the pan we determine the load Q^* at which the bloc starts moving. Obviously, the limiting friction $F_l = Q^*$ Hence, as in this case $N = P$, we find from Eq.(1.5.1)

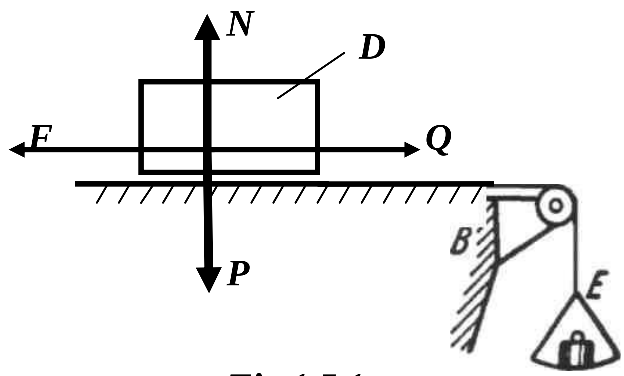


Fig.1.5.1

$$f = \frac{F_l}{N} = \frac{Q.^*}{P}$$

A series of such experiment demonstrates that, within certain limits, Q^* is proportional to P and f is constant. The coefficient of static friction is also independent of the magnitude of the contact area, within certain limits.
It should be noted that as long as the block remains at rest, the frictional force is equal to the applied force, and not to F_l. The force of friction becomes equal to fN only when slipping is impending
The foregoing refers to sliding friction of rest. When motion occurs, the frictional force is directed opposite to the motion and equals the product of kinetic coefficient

of friction and the normal pressure.

1.5.2. Angle of Friction. Reactions of Rough Constraints

Up till now we regarded the surfaces of constraints as smooth. The reaction of real (rough) constraints consists of two components: the normal reaction $\boldsymbol{N}$ and the frictional force $\boldsymbol{F}$ perpendicular to it. Consequently, the total reaction $\boldsymbol{R}$ forms an angle with the normal to the surface. As the friction increases from zero to F_l, force $\boldsymbol{R}$ changes from $\boldsymbol{N}$ to $\boldsymbol{R}_l$, its angle with the normal increasing from zero to a maximum value φ_0 (Fig. 1.5.2a). The maximum angle φ_0 which the total reaction of a rough support makes with the normal to the surface is called the angle of static friction, or angle of repose.

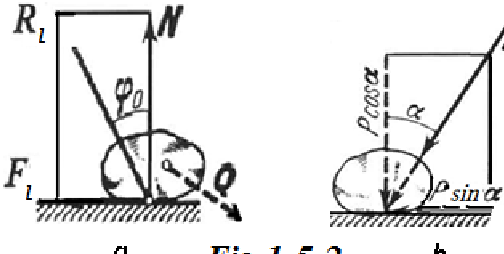


Fig.1.5.2

From Fig. 1.5.2a we obtain: $tg\varphi_0 = \frac{F_l}{N}$.
Since $F_l = fN$, we have the following relation between the angle of friction and the coefficient of friction:

$$tg\varphi_0 = f. \tag{1.5.2}$$

When a system is in equilibrium the total reaction $\boldsymbol{R}$ can pass anywhere within the angle of friction, depending on the applied forces. When motion impends, the angle between the reaction and the normal is φ_0.
If to a body lying on a rough surface is applied a force $\boldsymbol{P}$ making an angle α with the normal (Fig. 1.5.2b), the body will move only if the shearing force $P\sin\alpha > F_l = fP\cos\alpha$ (neglecting the weight of the body). But the inequality $P\sin\alpha > fP\cos\alpha$, where $f = tg\varphi_0$, is satisfied only if tan $\alpha > tg\varphi_0$, i.e., if $\alpha > \varphi_0$. Consequently, if angle α is less than φ_0 the body will remain at rest no matter how great the applied force. This explains the well-known phenomena of wedging and self-locking.

1.5.3. Belt Friction

A force $\boldsymbol{P}$ is applied at the end of a string passing over a cylindrical shaft (Fig. 1.5.3). Let us determine the least force $\boldsymbol{Q}$ that must be applied at the other end of the string to maintain equilibrium at a given angle $AOB = \alpha$ Consider the equilibrium of an element DE of the string of length $dl = Rd\theta$, where R is the radius of the shaft. The difference dT between the tensions in the string at D and E is balanced by the frictional force $dF = fdN$.

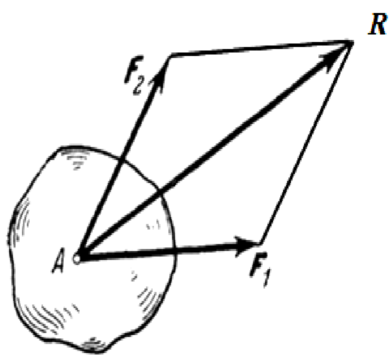


Fig.1.5.

Consequently, $dF = fdN$. The value of dN is determined from the equilibrium equation derived for the force components parallel to axis *Oy*.
Taking into account that for very small angles, $\sin\theta \approx \theta$, and neglecting small quantities of higher order we obtain:

$$dN = T\sin\frac{d\theta}{2} + (T + dT)\sin\frac{d\theta}{2} = 2T\frac{d\theta}{2} = Td\theta.$$

Substituting this value of dN, we have:

$$dT = fTd\theta.$$

Dividing both members of the equation by T and integrating the right-hand member in the interval from O to X and the left-hand member from Q to P, we have:

$$\int_Q^P \frac{dT}{T} = f\int_0^\alpha d\theta \text{ or } \ln\frac{P}{Q} = f\alpha,$$

whence

$$\int_Q^P \frac{dT}{T} = f\int_0^\alpha d\theta \text{ or } \ln\frac{P}{Q} = f\alpha. \qquad (1.5.3)$$

It follows from Eq.(1.5.3) that the required force Q depends only on the coefficient of friction f and the angle α. It does not depend on the radius of the shaft. If there is no friction $Q = P$. Of great practical importance is the fact that by increasing angle α (wrapping the string around the shaft) it is possible substantially to reduce the force $\boldsymbol{Q}$ required to balance force $\boldsymbol{P}$ .Eq. (1.5.3) also gives the relation between the tensions in the driving part and the driven part of a belt uniformly rotating a pulley without slippage.

1.5.4. Rolling Friction and Pivot Friction

Rolling friction is defined as the resistance offered by a surface to a body rolling on it. Consider a roller of radius R and weight P resting on a rough plane (Fig.1.5.4a). If we apply to the axle of the roller a force $Q < F_l$, there will be developed at A a frictional force $\boldsymbol{F}$ equal in magnitude to Q, which prevents the roller from slipping on the surface. If the normal reaction $\boldsymbol{N}$ is also assumed to be applied at A, it will balance force $\boldsymbol{P}$, with forces $\boldsymbol{Q}$ and $\boldsymbol{F}$ making a couple which turns the roller. If these assumptions were correct, we could expect roller to move, howsoever small the force $\boldsymbol{Q}$.

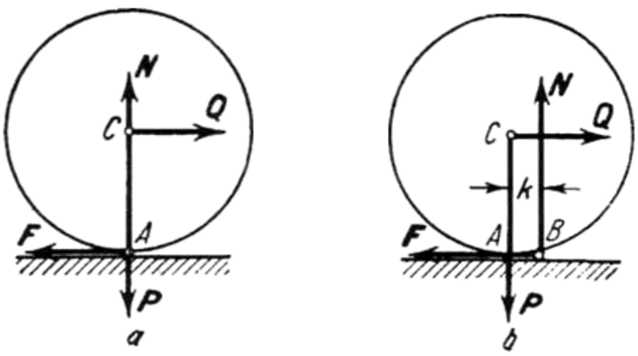


Fig.1.5.4

Experience tells us, however, that this is not the case; for, due to deformation, the bodies contact over a certain surface AB (Fig.1.5.4b). When force $\boldsymbol{Q}$ acts, the pressure at A decreases and at B increases. As a result, the reaction $\boldsymbol{N}$ is shifted in the direction of the action of force $\boldsymbol{Q}$. As, Q increases, this displacement grows till it reaches a certain limit k.

Thus, in the position of impending motion acting on the roller will be a couple $(\boldsymbol{Q}, \boldsymbol{F})$ with a moment Q_l, R balanced by a couple $(\boldsymbol{N}, \boldsymbol{P})$ of moment Nk. As the moments are equal, we have $Q_l R = Nk$ or

$$Q_l = \frac{k}{R}N. \qquad (1.5.4)$$

As long as $Q < Q_l$, the roller remains at rest; when $Q > Q_l$ it starts to roll. The linear quantity k in Eq. (1.5.4) is called the coefficient of rolling friction, and is generally measured in centimeters. The value of k depends on the material of the bodies and is

determined experimentally.
Consider a sphere at rest on a horizontal plane. If a horizontal couple with a moment M is applied to the sphere it will be tend to rotate it about its vertical axis. We know from experience that the sphere will start turning only when M exceeds some specific value M_l which is determined by the formula

$$M_l = \lambda N,$$

where N is the normal pressure of the sphere on the surface. This result is explained by the development of so-called pivot friction, i.e., resistance to rotation to the friction of the sphere on the surface. The factor λ is a linear quantity called the coefficient of pivot friction.

1.5.5. Illustrative problems

Problem 1. A load of weight $P = 100\,N$ rests on a horizontal surface (Fig.1.5.5). Determine, the force Q that should be applied at an angle $\alpha = 30°$ to the horizontal to move the load from its place, if the coefficient of static friction for the surfaces of contact is $f_0 = 0.6$.

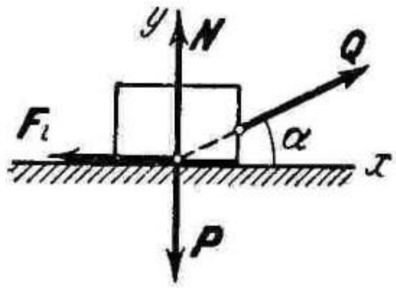


Fig. 1.5.5

Solution. According to the conditions of the problem we have to consider the position of impending motion of the load. In this position acting on it are forces $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{N}$ and $\boldsymbol{F}_l$. Writing the equilibrium equations in terms of the projections on the coordinate axes, we obtain:

$Q\cos\alpha - F_l = 0; \quad N + Q\sin\alpha - P = 0.$

From the second equation $N = P - Q\sin\alpha$, whence:

$$F_l = f_0 N = f_0(P - Q\sin\alpha).$$

Substituting this value of F_l in the first equation, we obtain finally:

$$Q = \frac{f_0 P}{\cos\alpha + f_0\sin\alpha} \approx 52\,N.$$

Problem 2. Determine the angle α to the horizontal at which the load on the inclined plane in Fig.1.5.6 remains in equilibrium if the coefficient of friction is f_0.

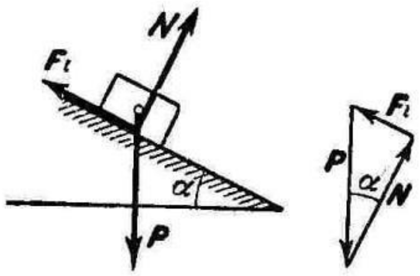


Fig. 1.5.6

Solution. The problem requires that all possible positions for the equilibrium of the load be determined. For this, let us first establish the position of impending motion at which $\alpha = \alpha_l$. In that position acting on the load are its weight $\boldsymbol{P}$, the normal reaction $\boldsymbol{N}$ and the limiting friction $\boldsymbol{F}_l$ Constructing a closed triangle with these forces, we find that $F_l = N\tan\alpha_l$. But, on the other hand, $F_l = f_0 N$. Consequently,

$$\tan\alpha_l = f_0.$$

In this equation α_l decreases as $f_0$decreases. We conclude, therefore, that equilibrium is also possible at $\alpha < \alpha_l$. Finally, all the values of α at which the load remains in equilibrium are determined by the inequality

$$\tan\alpha \le f_0.$$

If there is no friction ($f_0 = 0$), equilibrium is possible only at $\alpha = 0$.

*Problem 3***.** A bent bar whose members are at right angles is constrained at A and B as shown in Fig.1.5.7. The vertical distance between A and B is h. Neglecting the weight of the bar, determine the thickness d at which the bar with a load lying on its horizontal member will remain in equilibrium regardless of the location of the load. The coefficient of static friction of the bar on the constraints is f_0.

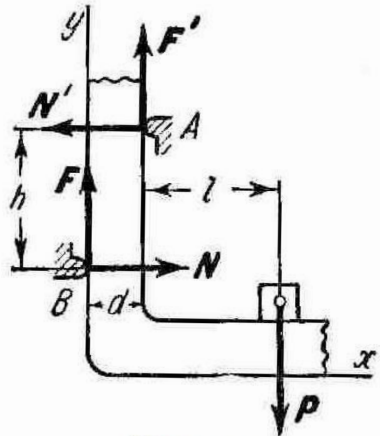


Fig. 1.5.7

Solution. Let us denote the weight of the load by P and its distance from the vertical member of the bar by l. Now consider the position of impending slip of the bar, when $d = d_l$. In this position acting on it are force $\boldsymbol{P}, \boldsymbol{N}, \boldsymbol{F}, \boldsymbol{N}'$, and $\boldsymbol{F}'$, where $\boldsymbol{F}$ and $\boldsymbol{F}'$ are the forces of limiting friction. Writing the equilibrium equations and taking the moments about A, we obtain:

$$N - N' = 0, \quad F + F' - P = 0, \quad Nh - Fd_l - Pl = 0,$$

where $F = f_0 N$ and $F' = f_0 N'$. From the first two equations we find:

$$N = N', \quad P = 2f_0 N.$$

Substituting these values in the third equation and eliminating N, we have:

$$h - f_0 d_l - 2f_0 l = 0,$$

whence

$$d_l = \frac{h}{f_0} - 2l.$$

If in this equation we reduce f_0 the right-hand side will tend to infinity. Hence, equilibrium is possible at any value of $d > d_l$. The maximum value of d_l is at $l = 0$. Thus, the bar will remain in equilibrium wherever the load is placed (at $l \ge 0$) if the inequality

$$d \ge \frac{h}{f_0}$$

is satisfied. The less the friction the grater must d be, If there is no friction ($f_0 = 0$) equilibrium is obviously impossible, as $d = \infty$.

Problem 4. Neglecting the weight of the ladder AB in Fig.1.5.8, determine the values of angle α at which a man can climb to the top of the ladder at B if the angle of friction for the contacts at the floor and the wall is φ_0.

Solution. Let us examine the position of impending slip of the ladder by graphical method. For impending motion the forces acting on the ladder are the reactions of the floor and wall $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$ which are inclined at the angle of friction φ_0 to the normals to the surfaces. The action lines of the reaction intersect at K. Thus, for the system to be in equilibrium the third force $\boldsymbol{P}$ (the weight of the man) acting on the ladder must also pass through K. Hence, in the position shown in the diagram the man cannot climb higher than D. For him to reach B the action lines of $\boldsymbol{R}_A$and $\boldsymbol{R}_B$ must intersect somewhere along BO, which is possible only if force $\boldsymbol{R}_A$ is directed along AB, i.e., when $\alpha \ll \varphi_0$.

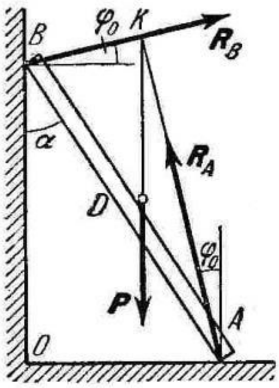


Fig. 1.5.8

Thus a man can climb to the top of a ladder only if its angle with the wall does not exceed the angle of friction with the floor. The friction on the wall is irrelevant, i.e., the wall may be smooth.

Problem. 5 A force $\boldsymbol{F}$ is applied to the lever DE of the band-brake in Fig.1.5.9. Determine the frictional torque M_T exerted on the drum of radius R, if $CD = 2C$ and the coefficient of friction of the band on the drum is $f_0 = 0.5$.

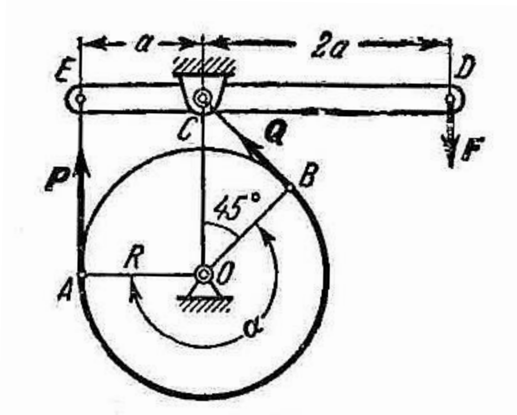


Fig.1.5.9

Solution. Acting on the drum and band AB wrapped around it is a force $\boldsymbol{P}$ (evidently $P = 2F$) applied at A and a force $\boldsymbol{Q}$ applied at B. We also have $f_0 = 0.5$ and $\alpha = \frac{5}{4}\pi = 3.93$ radians. Hence,

$$Q = 2Fe^{-\frac{5}{8}\pi} \approx 0.28\, F.$$

The required torque is

$$M_T = (P - Q)R = 1.72FR\,.$$

The less the value of Q i.e. the grater the coefficient of friction $f_0$and the angle α, the greater the torque.

Problem. 6 Determine the values of angle α at which a cylinder of radius R will remain at rest on an inclined plane if the coefficient of rolling friction is k (Fig.1.5.10).

Solution. Consider the position of impending motion, when $\alpha = \alpha_l$. Resolving force $\boldsymbol{P}$ into rectangular components $\boldsymbol{P}_1$and $\boldsymbol{P}_2$, we find that the moving force $Q_l = P_1 = P\sin\alpha_l$, and the normal reaction $N = P_2 = P\cos\alpha_l$. We have:

$$P\sin\alpha_l = \frac{k}{R}P\cos\alpha_l,$$

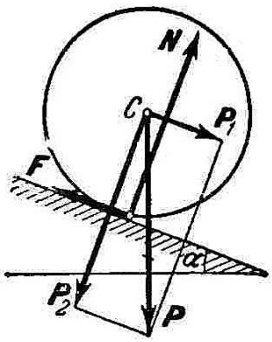


Fig. 1.5.10

or

$$\tan\alpha_l = \frac{k}{R}.$$

If k tends to zero the value of α_l also tends to zero. We conclude from this that equilibrium is maintained at any angle $\alpha < \alpha_l$.

1.6. FORCE SYSTEM IN SPACE

1.6.1. Vector Expression of the Moment of a Force about a Centre

The moment of a force about a centre as a measure of the tendency of that force to turn a body is characterized by the following three elements: 1) magnitude of the moment, which is equal to the product of the force and the moment arm; 2) the plane of rotation OAB through the line of action of the force and the centre (Fig.1.6.1); and 3) the sense of the rotation in that plane.

Fig.1.6.1

When all the given forces and the centre O are coplanar there is no need specify the plane of rotation, and the moment can be defined as a scalar algebraic quantity. If, however, the given forces are not coplanar, the planes of rotation have different aspects for different forces and have to be specified additionally. The position of a plane in space can be specified by vector normal to it. If, furthermore, the modulus of this vector is taken as representing the magnitude of the force moment, and the direction of the vector is made to denote the sense of rotation, such a vector completely specifies the three elements which characterize the moment of a force with respect to a given centre.

Thus, in general case we shall denote the moment $\boldsymbol{M}_0(\boldsymbol{F})$ of a force $\boldsymbol{F}$ about a centre O by a vector $\boldsymbol{M}_0$ applied at O, equal in magnitude to the product of the force $\boldsymbol{F}$ and the moment arm h, and normal to the plane OAB trough O and $\boldsymbol{F}$ We shall direct vector $\boldsymbol{M}_0$ so that the rotation viewed from the arrowhead is observed as counterclockwise .

Let's consider expression of moment of a force in terms of a vector product. From the definition,

$$|\boldsymbol{r} \times \boldsymbol{F}| = 2 \ \times \text{areas of} \vartriangle OAB = M_0$$

as vector $\boldsymbol{M}_0$ is equal in magnitude to twice the area of triangle OAB. Vector $(\boldsymbol{r} \times \boldsymbol{F})$ is perpendicular to plane OAB in the direction from which a counterclockwise rotation would be seen to carry $\boldsymbol{r}$ into $\boldsymbol{F}$ through the smaller angle between them, i.e., it is in the same direction as vector $\boldsymbol{M}_0$. Hence, vectors $(\boldsymbol{r} \times \boldsymbol{F})$ and $\boldsymbol{M}_0$ are equal in magnitude and direction. Therefore,

$$\boldsymbol{M}_0 = (\boldsymbol{r} \times \boldsymbol{F}). \qquad (1.6.1)$$

Thus, the moment of a force about a centre is equal to the vector product of the radius vector from the centre to the point of application of the force, and the force itself.

Formula (1.6.1) can be used to compute the moment $\boldsymbol{M}_0$ analytically. Suppose the projections F_x, F_y, F_z of force $\boldsymbol{F}$ on the axes and the x, y, z coordinates of its point of application A are known. Then, as $r_x = x,\ r_y = y$, and $r_z = z$, from the well-known formula we have

$$M_0 = \boldsymbol{r} \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & z \\ F_x & F_y & F_x \end{vmatrix}, \tag{1.6.2}$$

where $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are the unit vectors on the coordinate axes. If the determinant in the right-hand part of the equation is expanded according to the first row, the factors of $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ will be equal to the projection M_x, M_y and M_z of vector $\boldsymbol{M}_0$ on the coordinate axes as $\boldsymbol{M}_0 = M_x\boldsymbol{i} + M_y\boldsymbol{j} + M_z\boldsymbol{k}$. Consequently,

$$M_x = yF_z - zF_y\,,\ \ M_y = zF_x - xF_z,\ \ M_z = xF_y - yF_x. \tag{1.6.3}$$

Using Eqs. (1.6.3) we can determine the vector $\boldsymbol{M}_0$ from the following formula:

$$M = \sqrt{M_x^2 + M_y^2 + M_z^2}. \tag{1.6.4}$$

1.6.2. Moment of a Force With Respect to an Axis

Introduce the concept of moment of a force about an axis. The moment of a force about an axis is the measure of the tendency of the force to produce rotation about that axis. Consider a body free to rotate about an axis z (Fig.1.6.2).

Let a force $\boldsymbol{F}$applied at A be acting on body and a plane xy be passing through point A normal to the axis z. We can resolve the force $\boldsymbol{F}$ into rectangular components $\boldsymbol{F}_z$ parallel to the z axis and $\boldsymbol{F}_{xy}$ in the plane xy. Obviously, force $\boldsymbol{F}_z$, being parallel to axis z, cannot turn the body about that axis. Thus, we find that the total tendency of force $\boldsymbol{F}$ to rotate the body is the same as that of its component $\boldsymbol{F}_{xy}$. We conclude, then, that

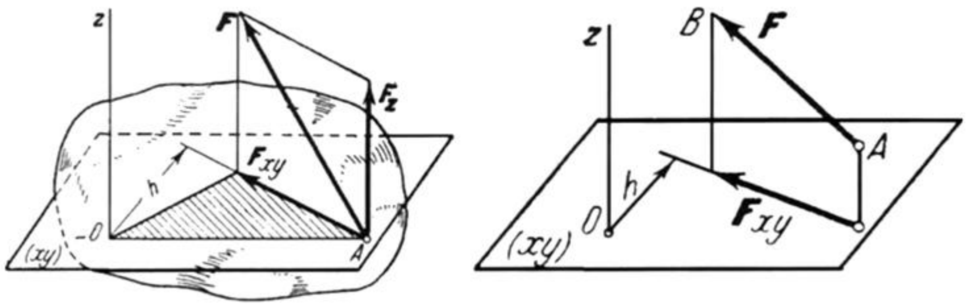


Fig.1.6.2

$$m_z(\boldsymbol{F}) = m_z(\boldsymbol{F}_{xy}),$$

where $m_x(\boldsymbol{F})$ is a moment of force $\boldsymbol{F}$ with respect to axis z. But the rotational effect of force $\boldsymbol{F}_{xy}$ is the product of the magnitude of this force and its distance h from the axis. On the other hand the moment of force $\boldsymbol{F}_{xy}$ with respect to point O, where the axis pierces the plane xy , is the same. Hence,

$$m_z(\boldsymbol{F}) = m_z(\boldsymbol{F}_{xy}) = m_0(\boldsymbol{F}_{xy}) = \pm F_{xy}h. \tag{1.6.5}$$

Thus, the moment of a force about an axis is an algebraic quantity equal to the moment of the projection of that force on a plane normal to the axis with respect to the point of intersection of the axis and the plane.

We shall call a moment positive if the rotation induced by a force $\boldsymbol{F}_{xy}$ is seen as

counterclockwise when viewed from the positive end of the axis, and negative if it is seen as clockwise.
In order to determine the moment of a force about axis z (Fig.1.6.2) we have to:

1) pass an arbitrary plane xy normal to the axis;
2) project force $\boldsymbol{F}$ on the plane and compute the magnitude of $\boldsymbol{F}_{xy}$;
3) erect a perpendicular from point O, where the plane and axis intersect, to the action line of $\boldsymbol{F}_{xy}$ and determine its length h;
4) compute the product $\boldsymbol{F}_{xy}h$;
5) determine the sense of the moment.

The following special cases should be borne in mind:
1) if a force is parallel to an axis, its moment about that axis is zero (since $\boldsymbol{F}_{xy} = 0$);
2) if the line of action of a force intersects with the axis, its moment with respect to that axis is zero (since $h = 0$);
Combining the two cases, we conclude that the moment of a force with respect to an axis is zero if the force and the axis are coplanar.

1.6.3. Relation between the Moments of a Force about a Centre and Axis

Consider a rectangular coordinate system with an arbitrary origin O and a force $\boldsymbol{F}$ applied at a point A whose coordinate are x, y, z (Fig.1.6.3). Let us determine analytically the moment of force $\boldsymbol{F}$ with respect to axis z. For this we project force $\boldsymbol{F}$ on the plane xy and resolve the projection into rectangular components $\boldsymbol{F}_x$ and $\boldsymbol{F}_y$. But, from the definition,
$m_z(\boldsymbol{F}) = m_0(\boldsymbol{F}_{xy}) = m_0(\boldsymbol{F}_x) + m_0(\boldsymbol{F}_y)$, which also follows Varignon's theorem. Also from the Fig.1.6.3
$m_0(\boldsymbol{F}_x) = -yF_x$, and $m_0(\boldsymbol{F}) = xF_y$.
Hence, $m_0(\boldsymbol{F}) = xF_y - yF_x$.

Fig.1.6.3

We can obtain the moments about the other two axes in the same way, and finally,

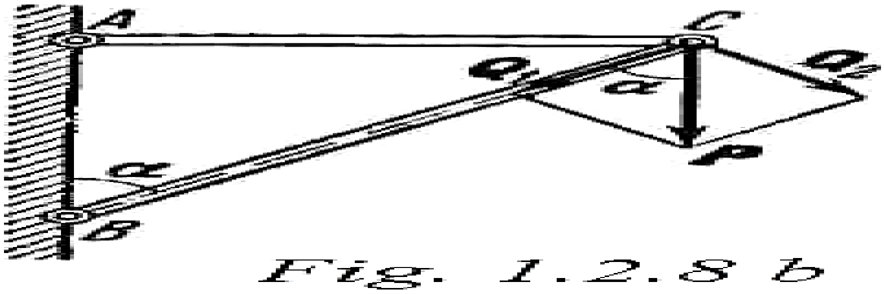
$$\left.\begin{aligned} m_x(\boldsymbol{F}) &= yF_z - zF_y, \\ m_y(\boldsymbol{F}) &= zF_x - xF_z, \\ m_z(\boldsymbol{F}) &= xF_y - yF_x \end{aligned}\right\}.$$

These equations give the analytical expression of the moments of a force about the axes of a Cartesian coordinate system.
Thus we can conclude that *the moment of a force with respect to an axis is equal to the projection on that axis of the vector denoting the moment of that force with respect to any point on the given axis.*

1.6.4. Composition of Couples in Space. Conditions of Equilibrium of Couples

The action of a couple on a body is characterized by the magnitude of the couple's

moment, the aspect of the plane of action, and the sense of rotation in that plane. In considering couples in space all three characteristics must be specified in order to define any couple. This can be done if, by analogy with the moment of a force, the moment of a couple is denoted by a vector whose modulus is equal to the magnitude of the couple's moment, normal to the plane of action in the direction from which the rotation would be observed as counterclockwise (Fig.1.6.4).
Since a couple may be located anywhere in its plane of action or in a parallel plane, it follows that vector m can be attached to any point of the body. Such vector is called a free one.



Fia.1.6.4

It is evident that vector $\boldsymbol{m}$ does, in fact, define the given couple as, if we know $\boldsymbol{m}$, by passing an arbitrary plane normal to $\boldsymbol{m}$, we obtain the plane of action of the couple; by measuring the length of $\boldsymbol{m}$ we obtain the magnitude of the couple moment; and the direction of $\boldsymbol{m}$ shows the sense of rotation of the couple.
Since a moment of couple in space is a vector value, couples in space are compounded according to the following theorem: *any system of couples acting on a rigid body is equivalent to a single couple with a moment equal to the geometrical sum of the moments of the component couples:*

$$\boldsymbol{M} = \sum_k \boldsymbol{m}_k \tag{1.6.6}$$

Vector $\boldsymbol{M}$ can be determined as the closing side of a polygon constructed with the component vectors as its sides.
If the component vectors are non-coplanar, the problem is best solved by the analytical method. From the theorem of the projection of a vector sum on an axis, and from Eq.(1.6.6), we obtain:

$$M_x = \sum_k m_{kx}, \quad M_y = \sum_k m_{xy}, \quad M_z = \sum_k m_{kx}. \tag{1.6.7}$$

With these projections we can construct vector $\boldsymbol{M}$. Its magnitude is given by the expression:

$$M = \sqrt{M_x^2 + M_y^2 + M_z^2}.$$

Any system of couples can be reduced to single couple with a moment determined by Eq.(1.6.6), but for equilibrium we must have $\boldsymbol{M} = 0$, or $\sum_k \boldsymbol{m}_k = 0$.
The analytical conditions of equilibrium can be found if we take into account that $\boldsymbol{M} = 0$ only if $M_x = 0$, $M_y = 0$ and $M_z = 0$. This, by virtue of Eqs.(1.6.7), is possible if

$$\sum_k m_{kx} = 0, \quad \sum_k m_{ky} = 0, \quad \sum_k m_{kz} = 0.$$

1.6.5. Reduction of a Force System in Space to a Given Centre

The problem of reducing an arbitrary force system to a given centre is based on the theorem of the translation of a force to a parallel position. In order to transfer a force $\boldsymbol{F}$ acting on a rigid body from a point A to a point O (Fig.1.6.5a.), we apply at O forces $\boldsymbol{F}' = \boldsymbol{F}$ and $\boldsymbol{F}" = -\boldsymbol{F}$. Force $\boldsymbol{F}' = \boldsymbol{F}$ will be applied at O together with the couple $(\boldsymbol{F}, \boldsymbol{F}")$ with a moment $\boldsymbol{m}$, which can also be shown as in Fig.1.6.5b.

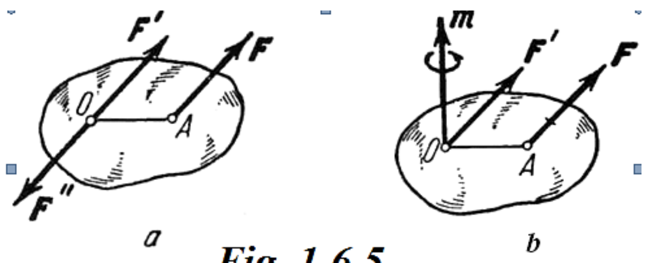


Fig. 1.6.5

We have: $\boldsymbol{m} = \boldsymbol{m}_0(\boldsymbol{F})$.

Consider now a rigid body on which an arbitrary system of forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \dots, \boldsymbol{F}_n$ is acting (Fig.1.6.6a.). Take any point O as the centre of reduction and transfer all the forces of the system to it, adding the corresponding couples. We have then acting on the body a system of forces

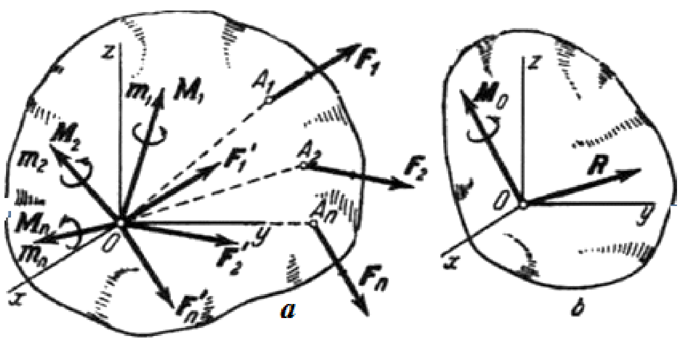


Fig.1.6.6

$$\boldsymbol{F}_1 = \boldsymbol{F}'_1,\ \ \boldsymbol{F}_2 = \boldsymbol{F}'_2, \dots,\ \ \boldsymbol{F}_n = \boldsymbol{F}'_n \tag{1.6.8}$$

applied at O and a system of couples whose moments are:

$$\boldsymbol{m}_1 = \boldsymbol{m}_0(\boldsymbol{F}_1),\ \ \boldsymbol{m}_2 = \boldsymbol{m}_0(\boldsymbol{F}_2), \dots \dots,\ \ \boldsymbol{m}_n = \boldsymbol{m}_0(\boldsymbol{F}_n).$$

The forces applied at O can be replaced by a single force $\boldsymbol{R}$ applied at the same point. This force is $\boldsymbol{R} = \sum_k \boldsymbol{F}'_k$, or, by Eq. (1.6.8),

$$\boldsymbol{R} = \sum_k \boldsymbol{F}_k. \tag{1.6.9}$$

We can compound all the obtained couples by geometrically adding the vectors of their moments. The system of couples will be replaced by a couple of moment $\boldsymbol{M}_0 = \sum_k \boldsymbol{m}_k$, or

$$\boldsymbol{M}_0 = \sum_k \boldsymbol{m}_0(\boldsymbol{F}_k). \tag{1.6.10}$$

The geometrical sum of all the forces is called the principal vector of the system. The geometrical sum of the moments of all the forces with respect to a given centre is called the principal moment of the system with respect to this centre.

Hence, we have proved the following theorem: *any system of forces can be reduced to an arbitrary centre and replaced by a single force, equal to the principal vector of the system applied at the centre of reduction, and a couple with a moment, equal to the principal moment of the system with respect to this centre* (Fig.1.6.6b).

Vector $\boldsymbol{R}$ and $\boldsymbol{M}_0$ are usually determined analytically, i.e., according to their projections on the coordinate axes:

$$R_x = \sum_k F_{kx},\ \ R_y = \sum_k F_{ky},\ \ R_z = \sum_k F_{kz}, \tag{1.6.11}$$

$$M_x = \sum_k m_x(\boldsymbol{F}_k),\ \ M_y = \sum_k m_y(\boldsymbol{F}_k),\ \ M_z = \sum_k m_z(\boldsymbol{F}_k). \tag{1.6.12}$$

It follows from the theorem that two systems of forces, for which $\boldsymbol{R}$ and $\boldsymbol{M}_0$ are the same, are statically equivalent. Hence, to define a force system it is sufficient to define its principal vector and its principal moment with respect to a given centre.

1.6.6. Conditions of Equilibrium of a Force System in Space

Reasoning as in section 1.4.3, we come to the conclusion that the necessary and sufficient conditions for the given system of forces to be in equilibrium are that $\boldsymbol{R} = 0$ and $\boldsymbol{M}_0 = 0$ But vectors $\boldsymbol{R}$ and $\boldsymbol{M}_0$ can be zero only if all their projections on the coordinate axes are zero, i.e., when $R_x = R_y = R_z = 0$ and $M_x = M_y = M_z = 0$, or, by Eqs.(1.6.11) and (1.6.12), when the acting forces satisfy the conditions:

$$\sum_k F_{kx} = 0, \quad \sum_k F_{ky} = 0, \quad \sum_k F_{kz} = 0,$$

$$\sum_k m_x(\boldsymbol{F}_k) = 0, \quad \sum_k m_y(\boldsymbol{F}_k) = 0, \quad \sum_k m_z(\boldsymbol{F}_k) = 0. \quad (1.6.13)$$

Thus, the necessary and sufficient conditions for the equilibrium of any force system in space are that the sums of the projections of all the forces on each of three coordinate axes and the sums of the moments of all the forces about those axes must separately vanish.

The first three of the equations express the conditions necessary for the body to have no translational motion parallel to the coordinate axes. The latter three equations express the conditions of no rotation about the axes.

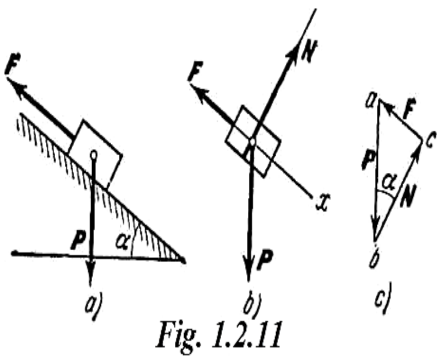


Fig.1.6.7

If all the forces acting on a body are parallel, the coordinate axes can be chosen so that the axis z is parallel to the forces (Fig.1.6.7). Then the x and y projections of all the forces will be zero, their moments about axis z will be zero, and the Eqs. (1.6.13) will be reduced to three conditions of equilibrium:

$$\sum_k F_{kx} = 0, \quad \sum_k m_x(\boldsymbol{F}_k) = 0, \quad \sum_k m_y(\boldsymbol{F}_k) = 0.$$

The other equations will turn into identities $0 \equiv 0$. Hence, the necessary and sufficient conditions for the equilibrium of a system of parallel forces in space are that the sum of the projections of all the forces on the coordinate axis parallel to the forces and the sums of the moments of all the forces about the other two coordinate axes must separately vanish.

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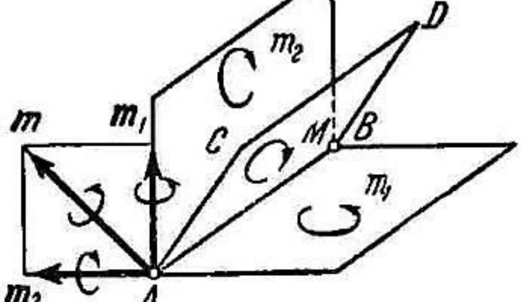

Fig.1.6.8

1.6.7. Illustrative problems

Problem 1. Acting on a rigid body are two couples in mutually perpendicular planes (Fig.1.6.8). The moment of each

is 30 *N-m*. Determine the resultant couple.
Solution. Denote the moments of the two couples by vectors $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ applied at an arbitrary point A; the moment of the resultant couples is denoted by vector $\boldsymbol{m}$. The resultant couple is located in plane $ABCD$ normal to $\boldsymbol{m}$ and the magnitude of the resultant moment is $30\sqrt{2}$ *N-m*.
If the sense of rotation of one of given couples is reversed, the resultant couple will occupy a plane normal to $ABCD$.
Problem. 2 The cube in Fig.1.6.9 hangs from two vertical rods AA_1 and BB_1 so that its diagonal AB is horizontal. Applied to the cube are couples $(\boldsymbol{P}, \boldsymbol{P}')$ and $(\boldsymbol{Q}, \boldsymbol{Q}')$. Neglecting the weight of the cube, determine the relation between forces P and Q at which it will be in equilibrium and the reactions of the rods.

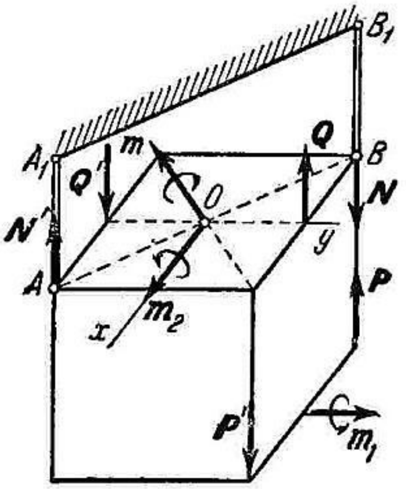

Fig. 1.6.9

Solution. The system of couples $(\boldsymbol{P}, \boldsymbol{P}')$ and $(\boldsymbol{Q}, \boldsymbol{Q}')$ is equivalent to a couple and can be balanced only by a couple. Hence, the required reactions $\boldsymbol{N}$ and $\boldsymbol{N}'$ must form a couple. Let us denote its moment $\boldsymbol{m}$ normal to diagonal AB as shown in the diagram. In scalar magnitude $m = Na\sqrt{2}$, where a is the length of the edge of the cube. Denote the moments of the given couples by the symbols $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$: their scalar magnitudes are $m_1 = Pa$ and $m_2 = Qa$ and their directions are as shown.
Now draw a coordinate system and write the equilibrium equations

$$\sum m_{kx} = m_2 - m\cos 45° = 0,$$
$$\sum m_{ky} = m_1 - m\cos 45° = 0.$$

The third condition is satisfied similarly.
It follows from the obtained equations that we must have $m_1 = m_2$, i.e., $Q = P$. We find, further, that

$$m = \frac{m_1}{\cos 45°} = m_1\sqrt{2} = Pa\sqrt{2}.$$

But $m = Na\sqrt{2}$, hence $N = P$.
Thus, equilibrium is possible when $Q = P$. The reactions of the rods are equal to P in magnitude and are directed as shown.
Problem. 3 Determine the stresses in section AA_1 of a beam subjected to forces as shown in Fig.1.6.10a. Force $\boldsymbol{Q}$ goes through the centre of the right-hand portion of the beam; force $\boldsymbol{F}$ lies in the plane Oxz; force $\boldsymbol{P}$ is parallel to the y axis.

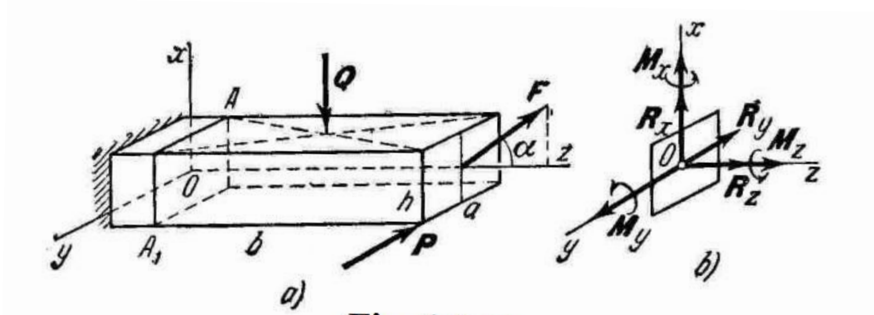

Fig. 1.6.10

Solution. Reduce all the forces to the centre O of the section, and place the origin of the coordinate system there. To determine the principal vector and principal moment of the system, we have:

$$R_x = F\sin\alpha - Q,$$
$$R_y = -P, \quad R_z = F\cos\alpha;$$
$$M_x = bP, \quad M_y = bF\sin\alpha - \frac{b}{2}Q, \quad M_z = \frac{h}{2}P.$$

Thus, acting on the section AA_1 are two lateral forces R_x and R_y, an axial tension R_z, and three couples of moments M_x, M_y, and M_z (Fig.1.6.10b): the first two tend to bend the beam about axes Ox and Oy and the last tends to twist it about axis Oz.

Problem. 4 Three workers lift a homogeneous rectangular plate whose dimensions are a by b (Fig.1.6.11). If one worker is at A, determine the points B and D where the other workers should stand so that they would all exert the same force.

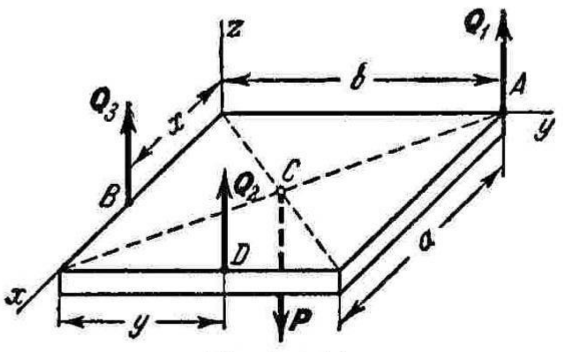

Fig. 1.6.11

Solution. The plate is a free body acted upon by four parallel forces $\boldsymbol{Q}_1, \boldsymbol{Q}_2, \boldsymbol{Q}_3$ and $\boldsymbol{P}$, where $\boldsymbol{P}$ is the weight of the plate. Assuming that the plate is horizontal and drawing the coordinate axes as shown in the figure, we obtain from the equilibrium conditions:

$$Q_1 b - Q_2 y - P\frac{b}{2} = 0,$$
$$-Q_2 a - Q_3 x + P\frac{a}{2} = 0,$$
$$Q_1 + Q_2 + Q_3 = P.$$

According to the conditions of the problem, $Q_1 = Q_2 = Q_3$, hence, from the last equation, $P = 3QP$. Substituting this expression in the first two equations and eliminating Q, we have:

$$b + y = \frac{3}{2}b, \qquad a + x = \frac{3}{2}a, \qquad x = \frac{a}{2}, \quad y = \frac{b}{2}.$$

Problem. 5 A horizontal shaft supported in bearings A and B as shown has attached at right angles to it a pulley of radius $r_1 = 0.2\,m$ and a drum of radius $r_2 = 0.15\ m$ (Fig.1.6.12). The shaft is driven by a belt passing over the pulley; attached to a cable wound on the drum is a load of weight $P = 1800\,N$ which is lifted with uniform motion when the shaft turns. Neglecting the weight of the construction, determine the reactions of the bearings and the tension T_1 in the

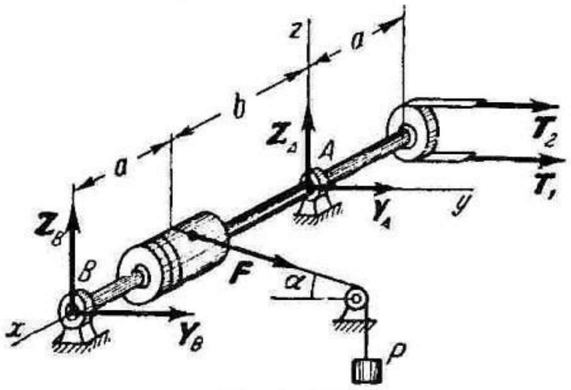

Fig. 1.6.12

driving portion of the belt, if it is known that, it is double the tension T_2 in the driven portion and if $a = 0.4\, m, b = 0.6\, m$, and $\alpha = 30°$.

Solution. As the shaft rotates uniformly, the forces acting on it are in equilibrium and the equations of equilibrium can be applied. Drawing the coordinate axes as shown and regarding the shaft as a free body, denote the forces acting on it: the tension $\boldsymbol{F}$ of the cable, which is equal to P in magnitude, the tensions $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ in the belt, and the reactions $\boldsymbol{Y}_A$, $\boldsymbol{Z}_A$, $\boldsymbol{Y}_B$, and $\boldsymbol{Z}_B$ of the bearings (each of the reactions $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ can have any direction in planes normal to the x axis and they are therefore denoted by their rectangular components).

From the equilibrium equations, and noting that $fF = P$, we obtain:

$$P\cos\alpha + T_1 + T_2 + Y_A + Y_B = 0,$$
$$-P\sin\alpha + Z_A + Z_B = 0,$$
$$-r_2P + r_1T_1 - r_1T_2 = 0,$$
$$bP\sin\alpha - (a+b)Z_B = 0,$$
$$bP\cos\alpha - aT_1 - aT_2 + (a+b)Y_B = 0.$$

Remembering that $T_1 = 2T_2$, we find immediately from the third and fourth equations that

$$T_2 = \frac{r_2P}{r_1} = 1350\, N,$$
$$Z_B = \frac{bP}{a+b}\sin\alpha = 540\, N.$$

From the fifth equation we obtain

$$Y_B = \frac{3aT_2 - bP\cos\alpha}{a+b} \approx 690\, N.$$

Substituting these values in other equations we find:

$$Y_A = -P\cos\alpha - 3T_2 - Y_B \approx -6300\, N,$$
$$Z_A = P\sin\alpha - Z_B = 360\, N,$$

and finally,

$$T_1 = 2700\, N,\ Y_A \approx -6300\, N, \qquad Z_A = 360\, N,\ Y_B \approx 690\, N,\ Z_B = 540\, N.$$

Problem 6. A rectangular plate of weight $P = 120\, N$ making an angle $\alpha = 60°$ with vertical is supported by a journal bearing at B and a step bearing at A (Fig.1.6.13a). The plate is kept in equilibrium by the action of a string DE; acting on the plate is a load $Q = 200\, N$ suspended from a string passing over pulley O and attached at K so that KO is parallel to AB. Determine the tension in string DE and the reactions of the bearings A and B, if $BD = BE$, $AK = a = 0.4\, m$, and $AB = b = 1\, m$.

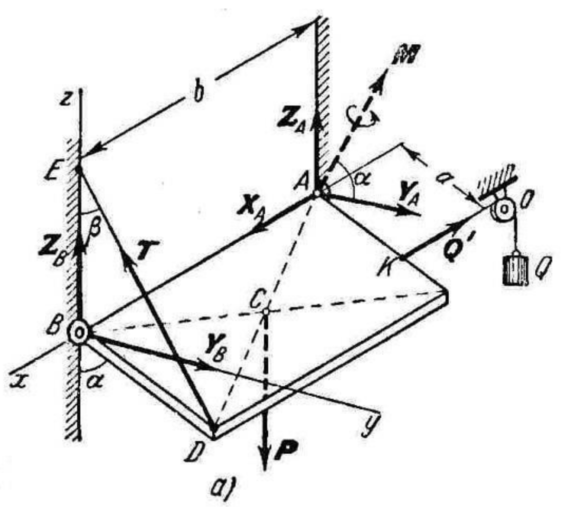

Fig. 1.6.13

Solution. Consider the equilibrium of the plate as a free body. Draw the coordinate axes with the origin at B (in which case force T intersects with the y and z axes, which simplifies the moment equations) and the acting forces and the reactions of the constraints as shown (the dashed vector M belongs to a different problem). For the equilibrium equations, calculate the projections and moments of all the forces; for this we introduce angle β and denote$d = BE = BD$. Computation of some of the moments is explained in the auxiliary diagrams (Figs1.6.13b and c). Fig.1.6.13b shows the projection on plane Byz from the positive end of the x axis. This diagram is useful in computing the moments of forces P and T about the x axis. It can be seen from the diagram that the projections of these forces on the yz plane are equal to the forces, and that the moment arm of force P with respect to point B is$BC_1 \sin\alpha = \frac{d}{2}\sin\alpha$; the moment arm of force $\boldsymbol{T}$ with respect to point B is $BE\sin\beta = d\sin\beta$.

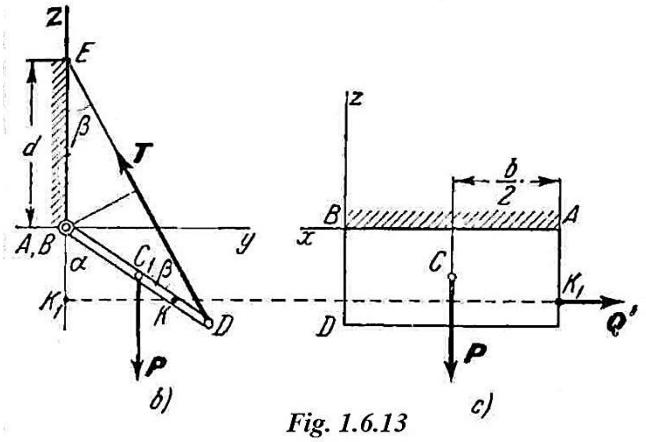

Fig. 1.6.13

Fig.1.6.13c shows the projection on plane Bxz from the positive end of the y axis. This diagram together with Fig.1.6.13b, helps to compute the moments of forces P and Q' about the y axis. It can be seen that the projections of these forces on the xz plane are equal to the forces themselves and that the moment arm of force P with respect to point B is $^1/_2\, AB = \frac{b}{2}$; the arm of force Q' with respect to B is AK_1, i.e.,

$AK\cos\alpha$, or $a\cos\alpha$, as is evident from Fig.1.13b.

Writing the equilibrium equations and assuming $Q' = Q$ we obtain:

$$-Q + X_A = 0,$$
$$-T\sin\beta + Y_A + Y_B = 0,$$
$$-P + T\cos\beta + Z_A + Z_B = 0,$$
$$-P\frac{d}{2}\sin\alpha + Td\sin\beta = 0,$$
$$-P\frac{b}{2} + Qa\cos\alpha + Z_A b = 0,$$
$$Qa\sin\alpha - Y_B b = 0.$$

Taking into account that $\beta = \frac{\alpha}{2} = 30°$, we find that

$$X_A = Q = 200\ N, \qquad T = P\frac{\sin\alpha}{2\sin\beta} \approx 104\ N,$$
$$Z_A = \frac{P}{2} - \frac{Qa}{b}\cos\alpha = 20\ N, \qquad Y_A = \frac{Qa}{b}\sin\alpha \approx 69\ N.$$
$$Y_B = T\sin\beta - Y_A = -170\ N, \quad Z_B = P - T\cos\beta - Z_A = 10\ N,$$

and finally,

$$T \approx 104\ N, \quad X_A = 200\ N, \quad Y_A \approx 69\ N,$$
$$Z_A = 20\ N, \quad Y_B = -17\ N, \quad Z_B = 10\ N.$$

Problem 7. Solve problem 7 for case when the plate is additionally subjected to a couple of moment $M = 120\, N \cdot m$ acting in the plane of the plate; the sense of rotation (viewed from the top of the plate) is counterclockwise.

Solution. Add to the forces in Fig.1.6.13a the moment vector $\boldsymbol{M}$ of the couple applied at any point perpendicular to the plate, e.g., point A. Its projections are: $M_x = 0\,, M_y = M\cos\alpha$, and $M_z = M\sin\alpha$. Applying the equilibrium equations, we find that the first and second equations remain the same as for problem 6 while the last two equations will be:

$$-P\frac{b}{2} + Z_A b + Qa\cos\alpha + M\cos\alpha = 0,$$
$$-Y_A b + Qa\sin\alpha + M\sin\alpha = 0.$$

Solving equations, we obtain results similar to those in problem 6, the only difference being that in all the equations Qa will be replaced by $Qa + M$. The answer is:

$$T \approx 104\, N,\quad X_A = 200\, N,\quad Y_A \approx 173\, N,\quad Z_A = -40\, N,$$
$$Y_B = -121\, N,\quad Z_B = 70\, N.$$

Problem 8. A horizontal rod AB is attached to a wall by a ball-and-socket joint and is kept perpendicular to the wall by wires KE and CD as shown in Fig.1.6.14a. Hanging from end B is a load of weight $P = 360\, N$. Determine the reaction of the ball-and-socket joint and the tensions in the wires if $AB = a = 0.8\, m$, $AC = AD_1 = b = 0.6\, m$, $AK = \frac{a}{2}, \alpha = 30°, and\ \beta = 60°$. Neglect the weight of the rod.

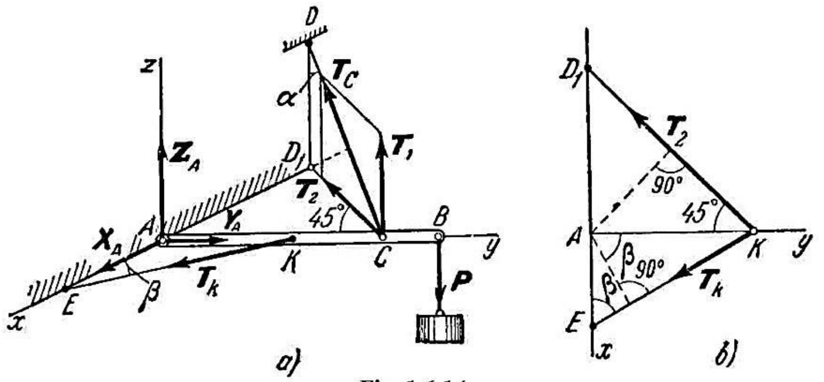

Fig. 1.6.14

Solution. Consider the equilibrium of the rod as a free body. Acting on it are force P and reactions $\boldsymbol{T}_k, \boldsymbol{T}_C, \boldsymbol{X}_A, \boldsymbol{Y}_A$, and $\boldsymbol{Z}_A$. Draw the coordinate axes and calculate the projections and moments of all the forces. As all forces pass through the y axis, their moments with respect to it are zero. To compute the moments of force $\boldsymbol{T}_C$ with respect to the coordinate axes, resolve it into components $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ $(T_1 = T_C\cos\alpha\,, T_2 = T_C\sin\alpha)$. We have $m_x(\boldsymbol{T}_C) = m_x(\boldsymbol{T}_1)$, as $m_x(\boldsymbol{T}_2) = 0$, and $m_z(\boldsymbol{T}_C) = m_z(\boldsymbol{T}_2)$, as $m_z(\boldsymbol{T}_1) = 0$. The computation of the moments of the forces with respect to the z axis is explained in the auxiliary Fig.1.6.14b showing the projection on plane Axy.

Substituting the values of T_1 and T_2 we obtain the following equations:

$$T_K\cos\beta - T_C\sin\alpha\sin 45° + X_A = 0,$$
$$-T_K\sin\beta - T_C\sin\alpha\cos 45° + Y_A = 0,$$
$$-P + T_C\cos\alpha + Z_A = 0,$$
$$-Pa + T_C b\cos\alpha = 0,$$

$$-T_K \frac{a}{2} \cos\beta + T_C b \sin\alpha \sin 45° = 0,$$

solving which we find that $T_C \approx 554\,N$, $T_K \approx 588\,N$, $X_A \approx -98\,N$, $Y_A \approx 705\,N$, and $Z_A = -120\,N$. Components X_A and Z_A thus actually act in the opposite direction than that shown in the diagram.

Problem 9. An equilateral triangular plate with sides of length a is supported in a horizontal plane by six bars as shown in Fig.1.6.15. Each inclined bar makes an angle of $\alpha = 30°$ with the horizontal. Acting on the plate is a couple of moment $\boldsymbol{M}$. Neglecting the weight of the plate, determine the stresses produced in the bars.

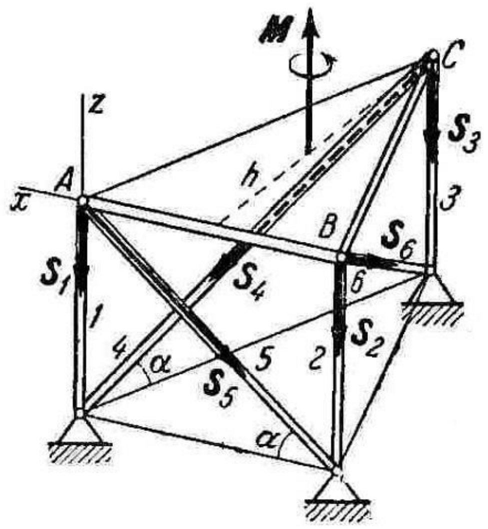

Fig. 1.6.15

Solution. Regarding the plate as a free body, draw, as shown in the figure, the vector of moment $\boldsymbol{M}$ of the couple and the reactions of the bars $\boldsymbol{S}_1, \boldsymbol{S}_2, \dots, \boldsymbol{S}_6$. Direct the reactions as if all the bars were in tension (i.e., we assume that the plate is being wrenched off the bars). As the body is in equilibrium, the sums of the moments of all the forces and couples acting on it with respect to any axis must be zero. Drawing axis z along bar *1* and writing the equations of the moment with respect to that axis, we obtain, as $M_z = M$,,

$$(S_6 \cos\alpha)h + M = 0,$$

where $h = \frac{a\sqrt{3}}{2}$ is the altitude of the triangle. From this we find:

$$S_6 = -\frac{2\sqrt{3}}{3}\frac{M}{a\cos\alpha}.$$

Writing the equations of the moments with respect to the axes along bars *2* and *3*, we obtain similar results for forces S_4 and S_5.

Now write the equations of the moments about axis x, which is directed alongside BA of the triangle. Taking into account that $M_x = 0$, we obtain

$$S_3 h + (S_4 \sin\alpha)h = 0,$$

whence, as $S_4 = S_6$, we find

$$S_3 = -S_4 \sin\alpha = \frac{2\sqrt{3}}{3}\frac{M}{a}\tan\alpha.$$

Writing the moment equations with respect to axes AC and CB, we obtain similar results for S_1 and S_2.

Finally, for $\alpha = 30°$, we have:

$$S_1 = S_2 = S_3 = \frac{2}{3}\frac{M}{a}; \quad S_4 = S_5 = S_6 = -\frac{4}{3}\frac{M}{a}.$$

The answer shows that the give couple creates tensions in the vertical bars and compressions in the inclined ones.

This solution suggests that it is not always necessary to apply equilibrium equations. There are several forms of equilibrium equations for non-coplanar force system, just as for coplanar systems.

In particular, it can be proved that the necessary and sufficient conditions for the equilibrium of a force system in space are that the sums of the moments of all the forces with respect to each of six axes directed along the edges of any triangular pyramid or along the side and base edges of a triangular prism are each zero.

The latter conditions were applied in solving the above problem.